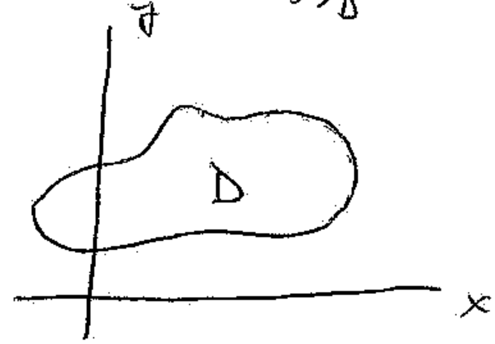
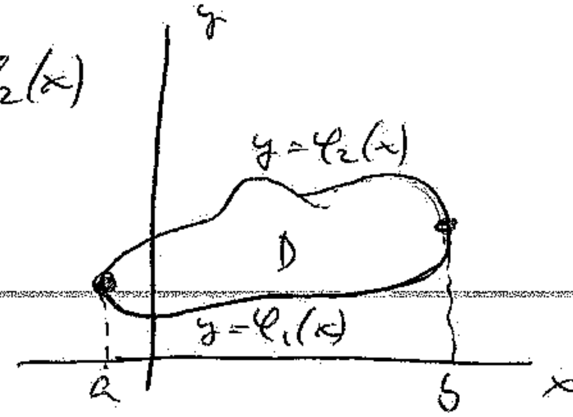
Lecture 25: [redacted]

Now, what if we needed to calculate $\iint_D f(x,y)\,dA$ over a region like here:



We can do two things:

① Find the extent of D in one dimension (say x) and then find expressions for 2 functions $y=\varphi_1(x)$ and $y=\varphi_2(x)$ so that



$$\iint_D f(x,y)\,dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$$

where we view D as y-simple.

<u>OR</u> ② Find a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which takes a <u>much</u> <u>nicer</u> <u>region</u> D^*, with coordinates (u,v) to D with coordinates (x,y).

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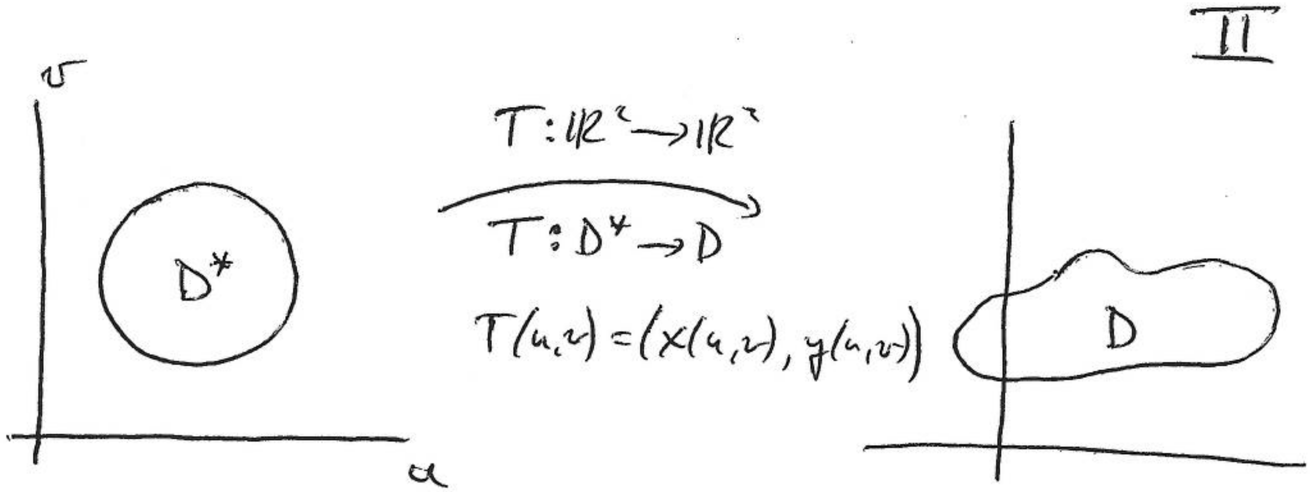

It turns out, changing the domain via a coordinate change can be quite advantageous!

Before studying integration, lets study these transformations.... in terms of some properties:

<u>Def</u> A map $T: D^* \to D$ is called <u>one-to-one</u> if whenever $T(u_1, v_1) = T(u_2, v_2)$, we have $u_1 = u_2$ and $v_1 = v_2$.

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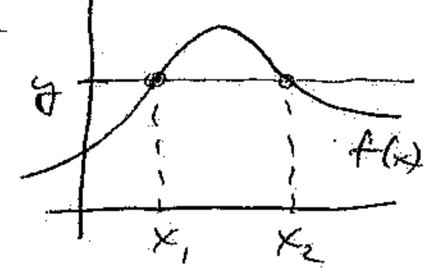
Note: Really, this means that no 2 pts in the domain are mapped to the same pt in the range. Recall in calculus I, the horizontal line test helped determine if f



was one-to-one (1-1). Here $f(x)$ is not one-to-one, since $f(x_1) = f(x_2) = y$ but $x_1 \neq x_2$.

<u>Def</u> A map $T: D^* \to D$ is called <u>onto</u> if $T(D^*) = D$. In other words, for every pt $(x,y) \in D$, there is at least one pt $(u,v) \in D^*$, where $T(u,v) = (x,y)$.

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ex. 1: The map $T:\mathbb{R}^2\to\mathbb{R}^2$, $T(r,\theta) = (r\cos\theta, r\sin\theta)$
$= (x,y)$

takes $D^* = [0,1]\times[0,2\pi]$ to
$D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1\}$

So $x = r\cos\theta$
$y = r\sin\theta$
here.

ex. 1
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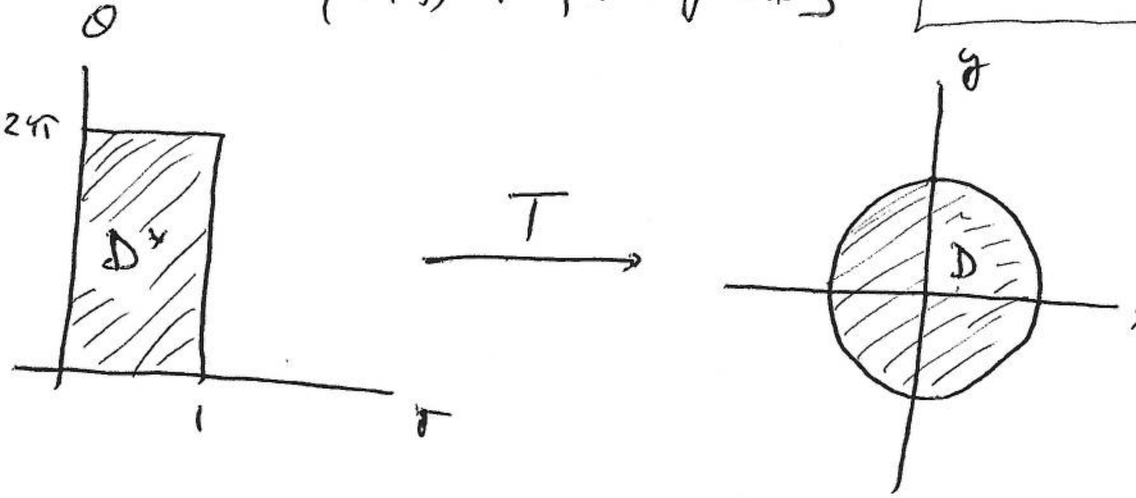
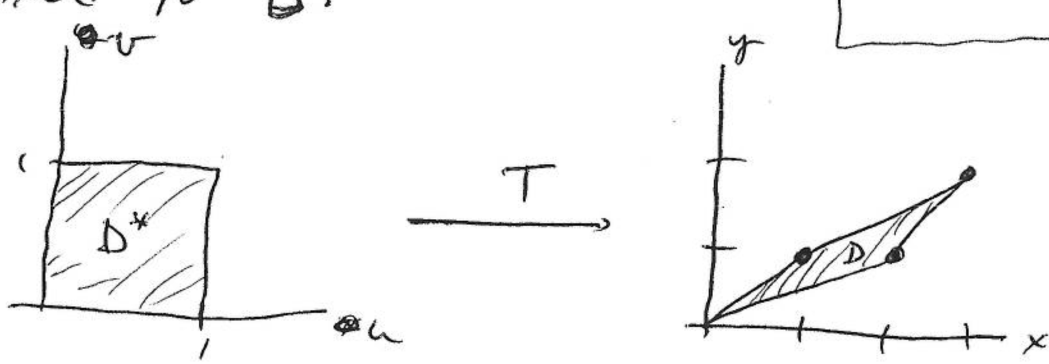


Here T is onto but not one-to-one.
(can you see why?) (where do the edges go?)

ex. 2: Let $T:\mathbb{R}^2\to\mathbb{R}^2$, $T(u,v) = (2u+v, u+v) = (x,y)$

Then the unit square in $\mathbb{R}^2$ gets mapped to D:

So $x = 2u+v$
$y = u+v$.



ex 2 (cont'd)

This T map is one-to-one: Suppose there are 2 inputs (u_1, v_1), (u_2, v_2) that both map to the same pt (x, y), so that

$$T(u_1, v_1) = (2u_1 + v_1, u_1 + v_1) = (2u_2 + v_2, u_2 + v_2)$$
$$= T(u_2, v_2).$$

Then $T(u_1, v_1) - T(u_2, v_2) = 0$ is the system

$$\left.\begin{aligned} 2u_1 + v_1 - 2u_2 - v_2 &= 0 \\ u_1 + v_1 - u_2 - v_2 &= 0 \end{aligned}\right\} \text{ or}$$

$$2(u_1 - u_2) + (v_1 - v_2) = 0$$
$$(u_1 - u_2) + (v_1 - v_2) = 0$$

Solving this linear system, we subtract the 2 eqns to get

$$\begin{array}{r} 2(u_1 - u_2) + (v_1 - v_2) = 0 \\ -\big((u_1 - u_2) + (v_1 - v_2) = 0 \big) \\ \hline (u_1 - u_2) + 0 = 0 \end{array} \Rightarrow u_1 = u_2.$$

and by the 2nd equation in the system,

$$(\cancel{u_1 - u_2}) + (v_1 - v_2) = 0 \Rightarrow v_1 = v_2.$$

ex. 2 (cont'd).

T is also onto here, since if $T(u,v)=(x,y)$

then $T(u,v)=(2u+v,u+v)=(x,y)$ is the

system $\begin{aligned} 2u+v&=x \\ u+v&=y. \end{aligned}$

Given any $(x,y)\in D$, we solve this system for (u,v):

~~[illegible]~~ $u=x-y\ ,\ v=2y-x.$

For any $(x,y)\in D$, there will be a $(u,v)\in D^*$ mapped to it. (Do you see this?)

Note: We can write this transformation

$$T: D^* \to D\ ,\ T(u,v)=(2u+v,u+v)=(x,y)$$

as $$T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)=A\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix},\ \text{where } A=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Note here in particular that $\det A = 2(1)-(1)1$
$= 1 \neq 0.$

T in this case is a linear transformation

<u>Thm</u> Let $A_{2\times 2}$ be nonsingular ($\det A \neq 0$), and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $T(\vec{x}) = A\vec{x}$

p.313 Then T is one-to-one, onto, takes parallelograms to parallelograms, vertices to vertices, and if D^* is a parallelogram, then so is $D = T(D^*)$.

Section 6.2

Back to the idea of integration: Suppose you were to calculate $\iint_D f(x,y)\,dA$ on some tricky domain D, and found a way to straighten out the domain via a transformation:

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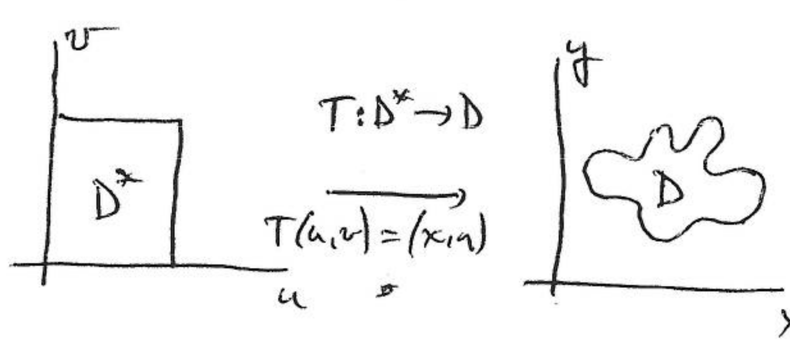
Here
$$T(u,v) = (x,y) = (x(u,v), y(u,v))$$



Then perhaps,

$$\iint_D f(x,y)\,dx\,dy \overset{?}{=} \iint_{D^*} f(x(u,v),y(u,v))\,du\,dv$$

If possible, then D^* may be a lot easier to integrate over.

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The problem is this may not work. Recall in Calc I the substitution method:

ex. Evaluate $\int_0^{\pi} 2x\sin(x^2)\,dx$. Switching to a new variable $u = x^2$ (in this case) also involved understanding how du is related to dx: $du = 2x\,dx$ (for $u=f(x)$, $du = f'(x)\,dx$)

We need to do the same in this multivariable sense here!

Def. For $T: D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ a C^1-transformation given by $T(u,v) = (x,y)$, where $x = x(u,v)$
$y = y(u,v)$,

the <u>Jacobian Determinant</u> of T is the determinant of the derivative matrix $DT(u,v)$.

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It is denoted

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

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ex. For $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$ we have $x = r\cos\theta$, $y = r\sin\theta$. Then

ex. 1 p. 315

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$
$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r.$$

ex. For $T: \mathbb{R}^2 \to \mathbb{R}^2$ $T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

here $x = 2u + v$, $y = u + v$.

Then $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1$$

Note: $\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = \det(A)$, for $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

We have the following:

<u>Thm</u> (Change of Variables formula).

Let D^* and D be elementary regions in $\mathbb{R}^2$ and let $T: D^* \to D$ be a C^1 transformation which is one-to-one and onto at least the interior of D^* and D. Then for $f: D \to \mathbb{R}$,

Thm 2 p. 319.

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

Note: ① This is precisely the 2-dimensional Substitution Method: Let $x = x(u,v)$
$y = y(u,v)$.

Then $dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

② These are absolute value signs. They are necessary and related to the transformation.

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③ For the transformation from polar to rectilinear: $x = r\cos\theta$, $y = r\sin\theta$, note that $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$.

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Thus $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$

Do you recognize this?