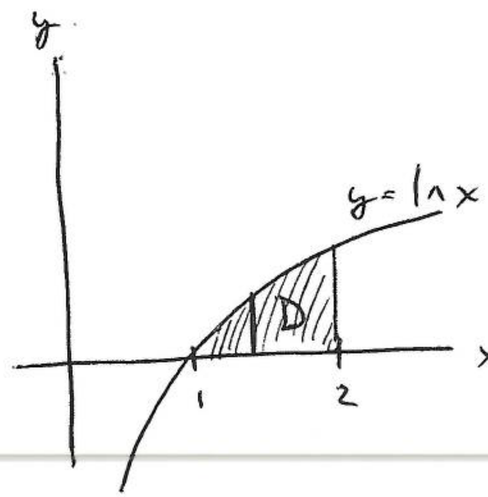
Lecture 24: [redacted]

Note: The previous example has a simple domain, (both x-simple and y-simple). Thus we can write the integrals using either.

Here is another example of a simple region ~~[illegible]~~ and how to switch from x-simple to y-simple.

ex 2 pg 291

Evaluate $\int_1^2 \int_0^{\ln x} (x-1)(\sqrt{1+e^{2y}})\, dy\, dx$



Here D is written as a y-simple region $(1 \le x \le 2,\ 0 \le y \le \ln x)$

Since the region is more complicated than a rectangle we cannot simply switch the nested integrals

Ex 2. p.291

$$\int_1^2 \int_0^{\ln x} (x-1)(\sqrt{1+e^{2y}})\, dy\, dx \neq \underbrace{\int_0^{\ln x} \int_1^2 (x-1)\sqrt{1+e^{2y}}\, dx\, dy}$$

choosing ~~an~~ y_0-slice between 0 and $\ln x$ in order to integrate $f(x, y_0)$ on $[1,2]$ doesn't make sense when y is a function of x.

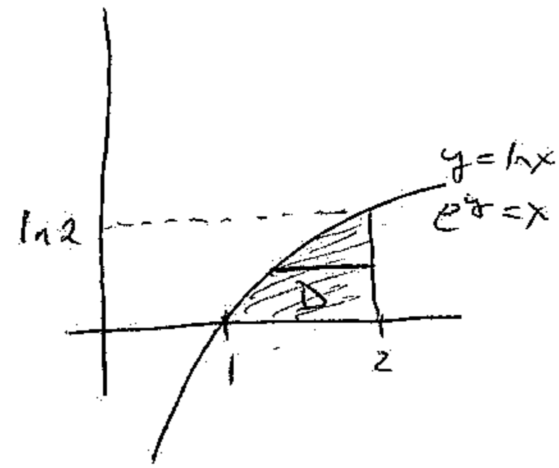
ex2 (cont'd)

If we want to "change the order of integration" in

$$\int_1^2 \int_0^{\ln x} (x-1)\sqrt{1+e^{2y}}\,dy\,dx$$

we need to view this y-simple region as an x-simple region:

$$0 \le y \le \ln 2, \quad e^y \le x \le 2.$$



The new format of the integral is

$$\int_0^{\ln 2} \int_{e^y}^{2} (x-1)\sqrt{1+e^{2y}}\,dx\,dy$$

Q: Why do this?

A: Sometimes the integral is tough one way and loads easier the other way.

Here $$\int_1^2 \int_0^{\ln x} (x-1)\sqrt{1+e^{2y}}\,dy\,dx = \int_1^2 (x-1)\left(\underbrace{\int_0^{\ln x}\sqrt{1+e^{2y}}\,dy}_{\text{How to do this?}}\right)dx$$

why can I pull this out?

XIV

$$\int_1^2 (x-1)\left(\int_0^{\ln x}\sqrt{1+e^{2y}}\,dy\right)dx \;\underset{\substack{\frac{u-1}{2}du=dy\\ y=0,\,u=2\\ y=\ln x,\,u=1+x^2}}{\overset{\substack{u=1+e^{2y}\\ du=2e^{2y}dy}}{=\!=\!=}}\; \int_1^2 (x-1)\left(\int_2^{1+x^2}\frac{u-1}{2}\sqrt{u}\,du\right)dx$$

$$=\int_1^2 (x-1)\left(\frac{1}{2}\int_2^{1+x^2}(u^{3/2}-u^{1/2})\,du\right)dx = \int_1^2 (x-1)\left(\frac{1}{2}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\Big|_2^{1+x^2}\right)dx$$

$$=\int (x-1)\left(\frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}-\frac{1}{5}\sqrt{32}+\frac{1}{3}\sqrt{8}\right)dx$$

Do we really want to continue this one??

Switching to an x-simple region, we get

$$\int_1^2 (x-1)\int_0^{\ln x}\sqrt{1+e^{2y}}\,dy\,dx = \int_0^{\ln 2}\int_{e^y}^{2}(x-1)\sqrt{1+e^{2y}}\,dx\,dy$$

$$=\int_0^{\ln 2}\sqrt{1+e^{2y}}\int_0^{\ln 2}(x-1)\,dx\,dy = \int_0^{\ln 2}\sqrt{1+e^{2y}}\left(\frac{x^2}{2}-x\Big|_{e^y}^{2}\right)dy$$

$$=\int_0^{\ln 2}\sqrt{1+e^{2y}}\left(2-2-\frac{1}{2}e^{2y}+e^y\right)dy$$

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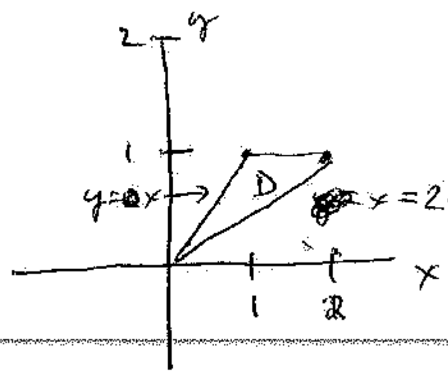
$$=\int_0^{\ln 2}\underbrace{e^y\sqrt{1+e^{2y}}\,dy}_{\substack{\text{use subst.}\\ u=e^y}} - \int_0^{\ln 2}\underbrace{\frac{1}{2}e^{2y}\sqrt{1+e^{2y}}\,dy}_{\substack{\text{use subst}\\ u=e^{2y}}}$$

= follow text pg 291.

ex. Change the order of integration and solve the integral both ways for

$$\int_0^1 \int_y^{2y} e^x \, dx \, dy$$

First, $D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} 0 \le y \le 1 \\ y \le x \le 2y \end{matrix} \right\}$ is x-simple



And $\int_0^1 \int_y^{2y} e^x \, dx \, dy$

$$= \int_0^1 \left(e^x \Big|_y^{2y} \right) dy = \int_0^1 \left(e^{2y} - e^y \right) dy$$

$$= \left(\tfrac{1}{2} e^{2y} - e^y \right) \Big|_0^1 = \tfrac{1}{2} e^2 - e - \left(\tfrac{1}{2} - 1 \right) = \tfrac{1}{2} e^2 - e + \tfrac{1}{2}.$$

Second, to write D as y-simple means acknowledging that the top function is $\varphi_2(x) = \begin{cases} x & 0 \le x \le \frac{1}{2} \\ 1 & 1 \le x \le 2 \end{cases}$, while $\varphi_1(x) = \frac{1}{2} x$. Can you see this? Hence

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} 0 \le x \le 2 \\ \varphi_1(x) \le y \le \varphi_2(x) \end{matrix} \right\}.$$

And then $\int_0^1 \int_y^{2y} e^x \,dx\,dy = \int_0^2 \int_{\frac{1}{2}x}^{\varphi_2(x)} e^x \,dy\,dx$

$$= \int_0^1 \int_{\frac{1}{2}x}^{x} e^x \,dy\,dx + \int_1^2 \int_{\frac{1}{2}x}^{1} e^x \,dy\,dx$$

Do you see why we broke this up into 2 pieces?

Thirdly, we evaluate this last set of integrals

$$= \int_0^1 \left(y e^x \Big|_{\frac{1}{2}x}^{x} \right) dx + \int_1^2 \left((y e^x) \Big|_{\frac{1}{2}x}^{1} \right) dx$$

$$= \int_0^1 \left(x e^x - \tfrac{1}{2} x e^x \right) dx + \int_1^2 \left(e^x - \tfrac{1}{2} x e^x \right) dx$$

$$= \int_0^1 \tfrac{1}{2} x e^x \,dx + \int_1^2 \left(e^x - \tfrac{1}{2} x e^x \right) dx$$

$$= \underbrace{\tfrac{1}{2} \int_0^1 x e^x \,dx - \tfrac{1}{2} \int_1^2 x e^x \,dx}_{} + \int_1^2 e^x \,dx$$

Careful: The limits are different as are the coefficients.
One cannot put these together easily.

Solution (cont'd)

$$\text{Here } \int_a^b x e^x \, dx \;\overset{\substack{u = x \quad v = e^x \\ du = dx \quad dv = e^x dx}}{\underset{\substack{\text{Int by} \\ \text{parts}}}{=\!=}}\; x e^x \Big|_a^b - \int_a^b e^x \, dx$$

$$= x e^x \Big|_a^b - e^x \Big|_a^b$$

$$\text{Hence } \frac{1}{2}\int_0^1 x e^x \, dx - \frac{1}{2}\int_1^2 x e^x \, dx + \int_1^2 e^x \, dx$$

$$= \frac{1}{2}\left(x e^x \Big|_0^1 - e^x \Big|_0^1 \right) - \frac{1}{2}\left(x e^x \Big|_1^2 - e^x \Big|_1^2 \right) + e^x \Big|_1^2$$

$$= \frac{1}{2}(e - e + 1) - \frac{1}{2}\left(2e^2 - e - (e^2 - e)\right) + e^2 - e$$

$$= \frac{1}{2} - e^2 + \frac{1}{2}e + \frac{1}{2}e^2 - \frac{1}{2}e + e^2 - e$$

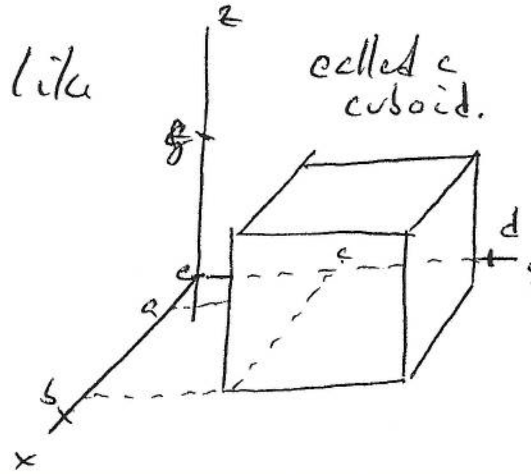
$$= \frac{1}{2} + \frac{1}{2}e^2 - e$$

which is the same as for the other way.

p.292 One more interesting fact: Mean-Value Thm for double integrals. We will skip this.

Q: What if we wanted to integrate a function $f(x,y,z)$ over a region in $\mathbb{R}^3$? If the region is a rectangular block, like

$$R = [a,b] \times [c,d] \times [e,f]$$



called a cuboid.

The 4-d volume formed by the graph $w = f(x,y,z)$ and the base xyz-space is given by the triple integral: $\iiint_R f(x,y,z)\,dV$

where dV is the product of dx, dy, and dz.

So $$\iiint_R f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$

p.295 $$= \int_c^d \int_e^f \int_a^b f(x,y,z)\,dx\,dz\,dy = \cdots$$

or other orderings of the nested integrals.

Like the 2-d rectangle, the rectangular box is a 3-d elementary region, where

* One of the variables is restricted to take values between 2 functions of the other 2 variables, where the domains of the 2 functions are all elementary in the plane.

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ex. The solid unit ball $x^2+y^2+z^2 \le 1$ is elementary, since we can describe this region as

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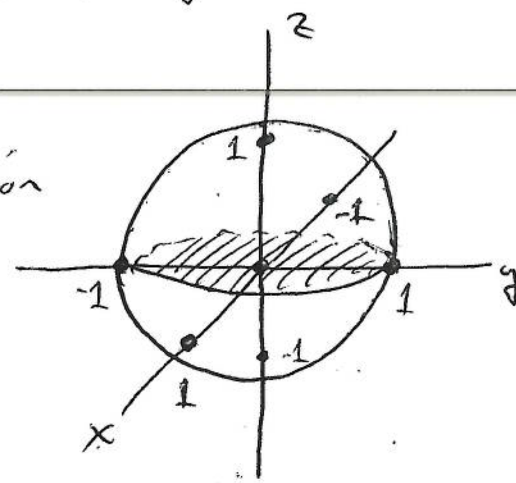
$$-1 \le x \le 1$$
$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$
$$-\sqrt{1-x^2-y^2} \le z \le \sqrt{1-x^2-y^2}$$



We can integrate any function on this ball as

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

In particular, we can integrate the function $f(x,y,z) = 1$ to recover the volume of the ball:

$$\text{volume (unit ball)} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(z \Big|_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \right) dy\,dx$$

$$= \int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\,dy\right) dx$$

$y = \sqrt{1-x^2}\sin u$

$dy = \sqrt{1-x^2}\cos u\,du$

when $y = -\sqrt{1-x^2}$, $u = -\frac{\pi}{2}$

$y = \sqrt{1-x^2}$, $u = \frac{\pi}{2}$

$$\int_{-1}^{1}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{(1-x^2)-(\sqrt{1-x^2}\sin u)^2} \cdot \sqrt{1-x^2}\cos u\,du\right) dx$$

$$= \int_{-1}^{1}\left(2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(1-x^2)(1-\sin^2 u)}\,\sqrt{1-x^2}\cos u\,du\right) dx$$

$$= \int_{-1}^{1} 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-x^2)\cos^2 u\,du\,dx = 2\int_{-1}^{1}(1-x^2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 u\,du\,dx$$

Solution: cont'd

and $$2\int_{-1}^{1}(1-x^2)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 u\,du\right)dx = 2\int_{-1}^{1}(1-x^2)\left[\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\cos 2u)\,du\right]dx$$

$$= 2\int_{-1}^{1}(1-x^2)\left(\frac{1}{2}\left(u+\frac{1}{2}\sin 2u\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)dx$$

$$= 2\int_{-1}^{1}(1-x^2)\left(\frac{1}{2}\left(\frac{\pi}{2}\right)+0-\frac{1}{2}\left(-\frac{\pi}{2}\right)+0\right)dx$$

$$= 2\int_{-1}^{1}(1-x^2)\frac{\pi}{2}\,dx = \pi\int_{-1}^{1}(1-x^2)\,dx = \pi\left(x-\frac{x^3}{3}\Big|_{-1}^{1}\right)$$

$$= \pi\left(1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right) = \frac{4}{3}\pi.$$

And the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

And beside the extra dimension, the manner of solving the triple integral is the same as that for a double integral.

ex. #35, pg 305

Find $\displaystyle\int_0^1\int_0^y\int_0^{x/\sqrt{3}} \frac{x}{x^2+z^2}\,dz\,dx\,dy$

<u>Strategy</u>: Here, we could directly integrate this

$$\int_0^1\int_0^y\int_0^{x/\sqrt{3}} x(x^2+z^2)^{-1}\,dz\,dx\,dy$$ with respect to z first,

using a trig substitution $z = x\tan u$. The rest follows.

<u>Solution</u>:

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$$\int_0^1\int_0^y\int_0^{x/\sqrt{3}} x(x^2+z^2)^{-1}\,dz\,dx\,dy \;\underset{\substack{z=\frac{x}{\sqrt{3}},\ u=\frac{\pi}{6}\\ z=0,\ u=0}}{\overset{\substack{z=x\tan u\\ dz=x\sec^2u\,du}}{=\!=\!=}}\; \int_0^1\int_0^y\left(\int_0^{\frac{\pi}{6}} x(x^2+x^2\tan^2u)^{-1}x\sec^2u\,du\right)dx\,dy$$

$$=\int_0^1\int_0^y\left(\int_0^{\frac{\pi}{6}} x^2(x^2\sec^2u)^{-1}\sec^2u\,du\right)dx\,dy$$

$$=\int_0^1\int_0^y \frac{\pi}{6}\,dx\,dy = \int_0^1\left(\frac{\pi}{6}x\Big|_0^y\right)dy = \int_0^1 \frac{\pi}{6}y\,dy = \left(\frac{\pi y^2}{12}\Big|_0^1\right) = \frac{\pi}{12}$$

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ex #27 pg 304 Set up the integral to find volume of

$$W = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le 1,\ z\ge 0,\text{ and } x^2+y^2+z^2\le 1\}.$$

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(Do you see that this is just the upper half of the unit ball?)

$$\mathrm{vol}(W) = \int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx.$$

#28 pg 304 Do the same for

$$W = \{(x,y,z)\in\mathbb{R}^3 \mid |x|\le 1, |y|\le 1, z\ge 0, \text{ and } x^2+y^2+z^2\le 1\}$$

This is a bit of a tricky one here since there are points of the first 2 constraints that cannot be satisfied by the third.

For example: $x=1=y$ does not satisfy $x^2+y^2+z^2<1$.

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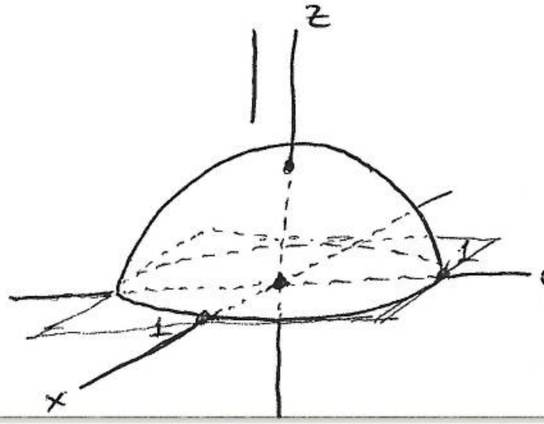

Here the xy plane constraints form the square $[-1,1]\times[-1,1]$ but the third constraint only works for $(x,y)\in\{x^2+y^2\le 1\}$.

the equator of the ball.

Here to find the volume, we only integrate exactly as in #27 pg 304.