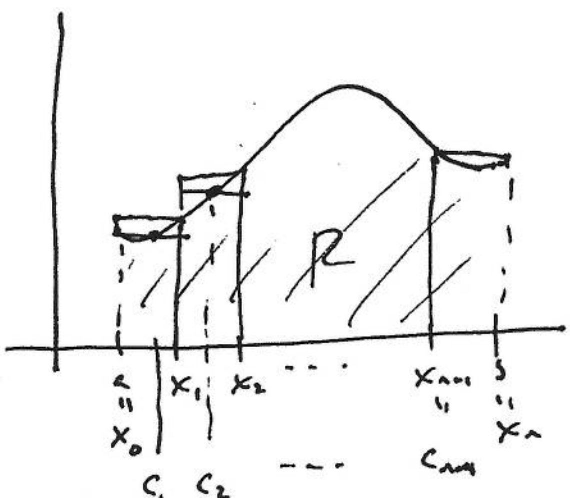
Lecture 22 [redacted] I

Recall a Riemann Sum: To approximate the area under a positive C^0-function $f:[a,b]\to\mathbb{R}$, we create a partition $a=x_0<x_1<\dots<x_n=b$ of $[a,b]$, and construct rectangles using $f(x_i)$ as the heights:



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Then area under $f(x)$ from a to b is approximated by

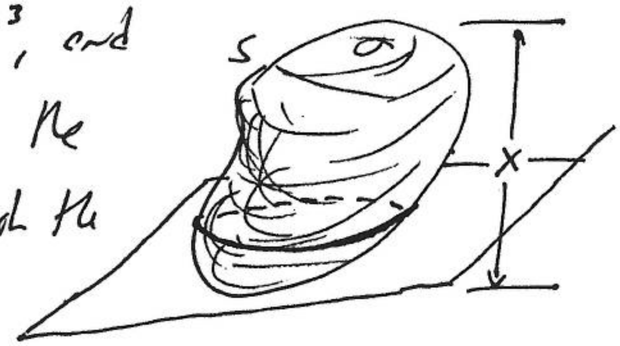
$$\text{area}(R) = \sum_{i=1}^{n} f(c_i)\,(x_i - x_{i-1})$$
$$= \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ and $c_i \in [x_{i-1}, x_i]$. This is a Riemann Sum, and the more larger the partition, the better the approx. In the limit,

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

In essence, $\int_a^b f(x)\,dx$ is the slicing up of $[a,b]$ into its pts $x\in[a,b]$, and then adding up all of the each x lengths of the lines from 0 to $f(x)$ at each x.

Cavalieri's Principle says the same is true for a solid S: ~~the~~ take a solid in $\mathbb{R}^3$, and along one direction through the solid, ~~where~~ x, slice through the solid by parallel planes P_x, as x ranges ~~from~~ on $[a,b]$.



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The area cut through the solid at x is $A(x)$.

Then volume $(S) = \int_a^b A(x)\,dx$

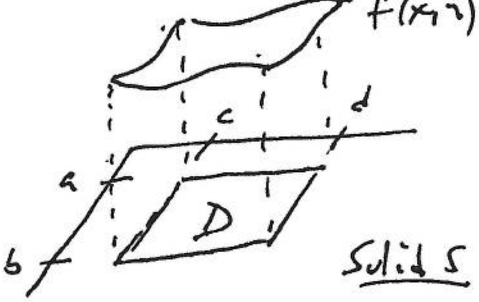
Note: One can build a Riemann Sum by slicing up the solid into thick slabs, and approximating the volume of each resulting disk , and adding up the resulting disk volumes.

Then take a limit as the thickness of each disk goes to 0. In the limit, the volume of each disk is (area of top) $\cdot\, dx = A(x)\,dx$.

Instead we will do the following:

Let ~~...~~ $f(x,y)$ be a positive function, whose graph has domain $[a,b]\times[c,d] = D$.

Then $z = f(x,y)$ forms a solid whose base is D and whose top is graph(f).



What is its volume? Approx. w/ a Riemann Sum:

Create partitions $a = x_0 < x_1 < \cdots < x_n = b$,

$$c = y_0 < y_1 < \cdots < y_n = d.$$

so that D is broken up into rectangles $\Delta A_{ij} = \Delta x_i \Delta y_j$

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and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$.

On each rectangle, choose any function height (p_i, q_j) and construct a solid block w/ base ΔA_{ij} and height $f(p_i,q_j)$

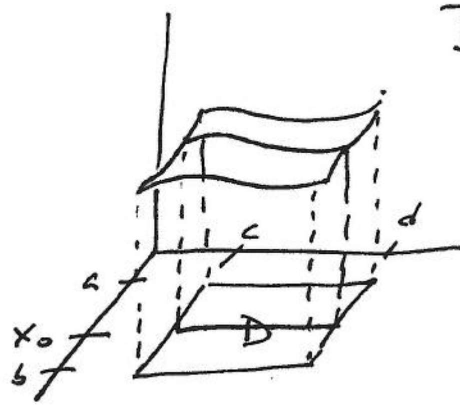
Then $\displaystyle \text{vol}(S) \cong \sum_{i,j=1}^{n} f(p_i,q_j)\,\Delta A_{ij} = \sum_{i,j=1}^{n} f(p_i,q_j)\,\Delta x_i \Delta y_j$

Larger partitions will give better approximations, so that

$$\text{vol}(S) = \lim_{\max \Delta A_{ij}\to 0} \sum_{i,j=1}^{n} f(p_i,q_j)\,\Delta A_{ij} = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

But what is this?

Back to Cavalieri's Principle:



If we fix an x_0 and slice through our solid at x_0 we get a 2 dim. figure

$f(x_0, y)$ on $[c,d]$.

its area is $\int_c^d f(x_0,y)\,dy$

If then we "add up" all of the x_0 slices on $[a,b]$ we get

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$$\text{vol}(S) = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

Of course, we can do this the other way also to get

$$\text{vol}(S) = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

In either case, we get the double integral of $f(x,y)$ over a rectangle $[a,b] \times [c,d]$ as

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

This also gives us a means to evaluate this integral!

(I) Integrate wrt one variable, leaving the other constant

(II) The result is a function of the other variable.

(III) Integrate wrt the other.

Notes: ① Fubini's Thm says that it does not matter which you integrate with first, as long as you are careful to "nest" the integrals

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$$\int_a^b \underbrace{\int_c^d f(x,y)\, dy}_{\text{together}}\, dx = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$

② In Calc I, discontinuous functions were integrable as long as ~~the~~ ⓐ discontinuities were jump disc. ⓑ there were a finite # of them.

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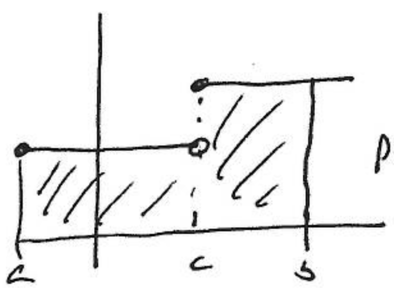

perfectly fine function to integrate.

$$\int_a^b f\, dx = \int_a^c f\, dx + \int_c^b f\, dx.$$

In Calc III, as long as the discontinuities are finite on each direction integral, you are fine.

ex Find $\iint_R x^2 y \, dA$, $R = [0,1] \times [1,2]$.

Here $\iint_R x^2 y \, dA \overset{①}{=} \int_0^1 \int_1^2 x^2 y \, dy \, dx$ (nested)

$\overset{②}{=} \int_1^2 \int_0^1 x^2 y \, dx \, dy$ (nested the other way).

① $$\int_0^1 \int_1^2 x^2 y \, dy \, dx = \int_0^1 \left(\int_1^2 x^2 y \, dy \right) dx = \int_0^1 \left(\frac{x^2 y^2}{2} \Big|_1^2 \right) dx$$

$$= \int_0^1 \left(\frac{4x^2}{2} - \frac{x^2}{2} \right) dx = \int_0^1 \frac{3}{2} x^2 \, dx$$

$$= \frac{1}{2} x^3 \Big|_0^1 = \frac{1}{2}(1) - \frac{1}{2}(0) = \frac{1}{2}.$$

② $$\int_1^2 \int_0^1 x^2 y \, dx \, dy = \int_1^2 \left(\int_0^1 x^2 y \, dx \right) dy = \int_1^2 \left(\frac{x^3 y}{3} \Big|_0^1 \right) dy$$

$$= \int_1^2 \frac{y}{3} \, dy = \frac{y^2}{6} \Big|_1^2 = \frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$ ∎