Lecture 21: [illegible] II

p. 245

Det For a vector field in $\mathbb{R}^n$, $\vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix} = F_1\vec{e}_1 + \cdots$
$\cdots + F_n\vec{e}_n$, the <u>divergence</u> of $\vec{F}$ is the <u>scalar field</u>

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$$

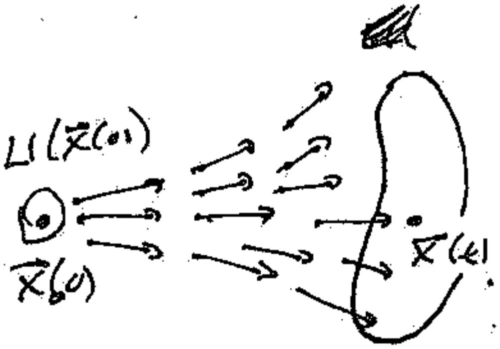
$$= \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$$

<u>Notes</u> ① A <u>scalar field</u> is just another name for a real-valued function (associated to each $\vec{x} \in \mathbb{R}^n$ a number, a scalar.

② In $\mathbb{R}^3$, $\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

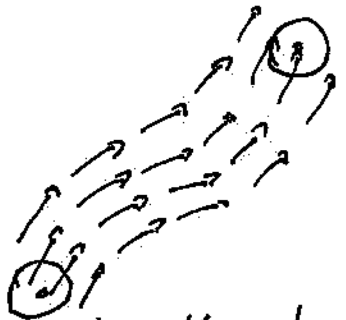
③ Divergence is a measure of the rate of expansion of volume at each pt in a vector field.

Think of it this way: At a given pt of a vector field with its attached vector, how are all of the nearby vectors oriented with respect to that one? Are they oriented in directions farther away, toward each other, or all in the same direction:



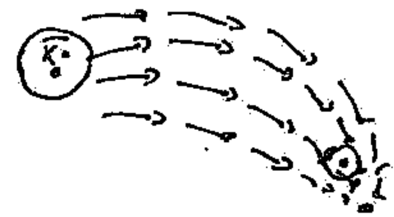
U after flowing along the vector field.

Here volume expands so $\operatorname{div}\vec{F}(\vec{x}_0) > 0$



Here the volume of U does not change over time.

Hence $\operatorname{div}\vec{F}(\vec{x}_0) = 0$



After flowing for a while volume of U decreases.

Hence $\operatorname{div}\vec{F}(\vec{x}_0) < 0$.

ex. Find the divergence of $\vec{F} = \begin{bmatrix} x^3 \\ x\sin(xy) \end{bmatrix}$. Interpret.

Here $$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x\sin(xy))$$
$$= 3x^2 + x^2\cos(xy))$$
$$= x^2(3+\cos(xy))$$

What can we say about this vector field?

① It is never contracting (compressing) since $\operatorname{div}\vec{F} \geq 0$ everywhere.

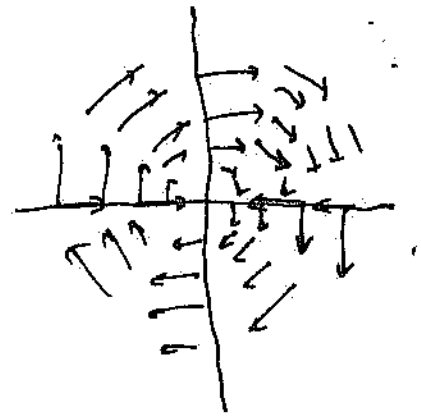
② Off the y-axis, it is always expanding since $\operatorname{div}\vec{F} > 0$ when $x \neq 0$.

Hence volumes will always expand under this vector field. (why?)

ex The divergence of $\vec{G} = \begin{bmatrix} y \\ -x \end{bmatrix}$ is 0 everywhere,

since $\operatorname{div} \vec{G} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) = 0$.

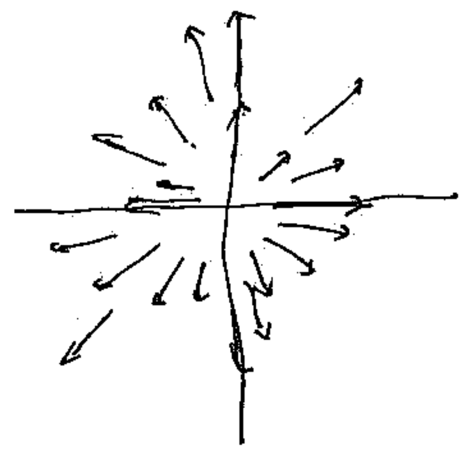
Can you see this vector field and what it does? Envision the path that would follow these vectors.



ex. The gradient field $\vec{H} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ comes from the function $h(x,y) = x^2 + y^2$.

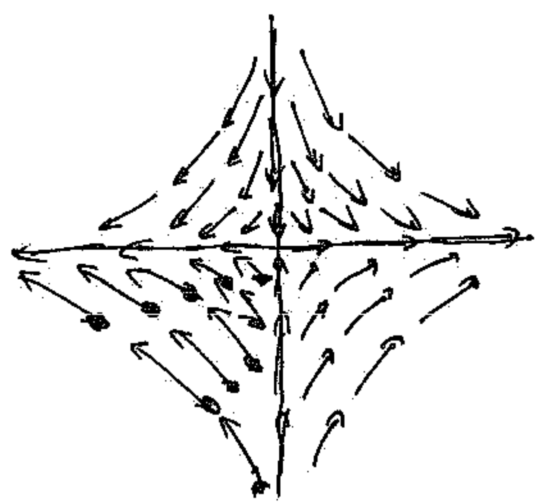
$\operatorname{div} \vec{H} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) = 4 > 0$

Does this make intuitive sense?



ex. How about $\vec{I} = \begin{bmatrix} x \\ -y \end{bmatrix}$?

Is this a gradient vector field? If so, what is the function? Is this compressing? expanding? Neither? A mix of each?



Another property of a vector field, but this one only in $\mathbb{R}^3$, is its curl:

Def For $\vec{F}(\vec{x}) = F_1(\vec{x})\vec{i} + F_2(\vec{x})\vec{j} + F_3(\vec{x})\vec{k} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$

a C^1-vector field, the curl of $\vec{F}$ is

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

p. 244.

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

ex. Calculate curl of $\vec{F} = \begin{bmatrix} y \\ -x \\ z \end{bmatrix}$. Explain what you "see"?

Solution: $\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z \end{vmatrix}$

$$= \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(-x)\right)\vec{i} - \left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(y)\right)\vec{j} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right)\vec{k}$$

$$= -2\vec{k}$$

Here the xy-plane (at $z=0$) simply rotates, while off this plane the vector field flow lines spiral away from the $z=0$ plane. The curl seems to reflect this rotation about the z-axis.

ex. Calculate the curl of the gradient field

$$\vec{G} = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}.$$ (What is the function that generates this gradient field?)

Solution: $\text{curl}\,\vec{G} = \nabla \times \vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$

$$= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right)\vec{i} - \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right)\vec{j} + \left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right)\vec{k}$$

$$= (x-x)\vec{i} - (y-y)\vec{j} + (z-z)\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Notes ① Notice that the curl of a vector field is another vector field. Recall the divergence of a vector was a scalar field (a real-valued function).

② The notation $\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$ is a matrix notation for "take the determinant of" $\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$.

③ Learn the patterns of the curl calculation: for example, in each component, only the partials of the other 2 variables show up.

④ There are more examples in the book.

ex (8, pg 249) The curl$(xy\vec{i} - \sin z\,\vec{j} + \vec{k})$
$= \cos z\,\vec{i} - x\vec{k}$.

Q: What does the curl actually measure?
A: A full explanation would not yet make sense.
We wait until chapter 8. But here is a hint:

Def. A vector field $\vec{F}$ whose curl $\vec{F} = \nabla \times \vec{F} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
is called irrotational.

p. 251

Hence a vector field is related to the twisting of a vector field near a particular vector.

---

ex. Suppose $\vec{F} = \nabla f$ is a gradient vector field.
for $f : \mathbb{R}^3 \to \mathbb{R}$.
Notice then that

$$\operatorname{curl}(\nabla f) = \nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

p. 252.

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} - \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k} = \vec{0}$$

since mixed partials are always equal.

Conclusion I: Gradient vector fields are ALWAYS irrotational (have no curl).

Conclusion II: If a vector field has non-zero curl, then it cannot be a gradient field!

ex. Notice from above:

① for $f(\vec{x}) = xyz$, $\vec{G} = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} = \nabla f$, $\text{curl}(\vec{G}) = \vec{0}$.

② for ex 8 pg 245, $\vec{H} = \begin{bmatrix} xy \\ -\sin z \\ 1 \end{bmatrix}$ is not a gradient vector field.

Hence we learned: For any real-valued C^1 function on $\mathbb{R}^3$, $f: \mathbb{R}^3 \to \mathbb{R}$, $\nabla \times \nabla f = \vec{0}$.

Now since the curl of any vector field is another vector field, what is its divergence?

That is, given $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a C^1-vector field, what is $\text{div}(\text{curl}(\vec{F})) = \nabla \cdot (\text{curl}\,\vec{F})$

$= \nabla \cdot (\nabla \times \vec{F})$?

We write it out: Let $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$. Then

$$\operatorname{curl}(\vec{F}) = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$= \begin{bmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{bmatrix}$$

p. 253.

Hence
$$\nabla \bullet (\operatorname{curl} \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 F_2}{\partial x \partial z}} + \cancel{\frac{\partial^2 F_1}{\partial y \partial z}} - \cancel{\frac{\partial^2 F_3}{\partial x \partial y}} + \cancel{\frac{\partial^2 F_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 F_1}{\partial z \partial y}}$$

$$0 \quad 0 \quad 0$$

$$= 0$$

<u>Conclusion</u>: The curl of a vector field is always divergence free!

<u>Conclusion</u>: Gradient vector fields are always curl free!

Conclusion: Nonzero div means not a curl field!

There are many more examples of vector analysis identities on pg. 255.

Go through these.