Lecture 20: [redacted]

<u>Def</u> A <u>vector field</u> is a map $\vec{F}: U \subset \mathbb{R}^n \to \mathbb{R}^n$

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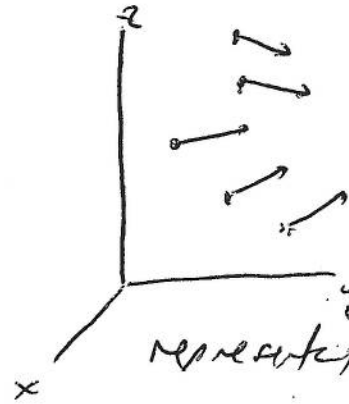
that assigns to each pt $\vec{x} \in U$, a vector $\vec{F}(\vec{x})$ based at $\vec{x}$ in $\mathbb{R}^n$..



representative vectors.

For $n=3$,

$$F(x,y,z) = \begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{bmatrix}$$

$$= F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}.$$

<u>examples</u> (1) Force fields in physics

(2) Slope or direction fields in ODEs.

<u>Special kind of vector field</u>

. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a C^1-function.

Then $\nabla f: \mathbb{R}^n \to \mathbb{R}^n$ can be viewed as a vector field, called the gradient field of f.

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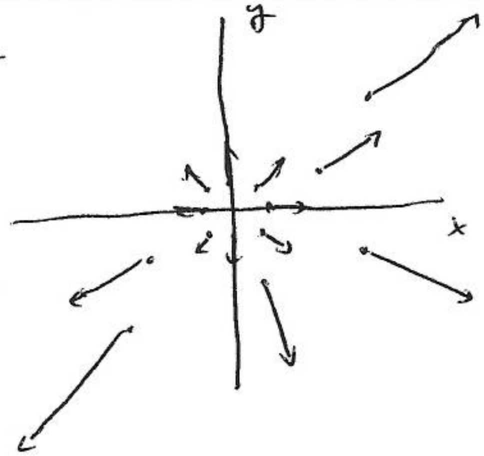
ex. let $f(x,y) = x^2 + y^2$

Then $\nabla f(x,y) = 2x\vec{i} + 2y\vec{j}$ is the gradient vector field



Note: Every C^1-function $f: \mathbb{R}^n \to \mathbb{R}$ has a gradient vector field.

Q: Can every vector field be seen as a gradient field for some function?

A: No.

ex: let $F(x,y) = y\vec{i} - x\vec{j} = \begin{bmatrix} y \\ -x \end{bmatrix}$.

If $F(x,y)$ were the gradient field of a function $f(x,y): \mathbb{R}^2 \to \mathbb{R}$, then $\frac{\partial f}{\partial x}(x,y) = y$, $\frac{\partial f}{\partial y}(x,y) = -x$.

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But for f to exist, the mixed partials must be equal:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \overset{\text{must}}{=\!=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial}{\partial y}(y) = 1 \neq -1 = \frac{\partial}{\partial x}(-x).$$

Hence there is no function $f: \mathbb{R}^2 \to \mathbb{R}$ where

$$\nabla f: \mathbb{R}^2 \to \mathbb{R}^2 \quad \nabla f(x,y) = \begin{bmatrix} y \\ -x \end{bmatrix}.$$

<u>Def.</u> A vector field $\vec{F}(\vec{x})$ on $\mathbb{R}^n$ is called <u>conservative</u> if there exists a function $V: \mathbb{R}^n \to \mathbb{R}$ where $\vec{F}(\vec{x}) = \nabla V(\vec{x})$.

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<u>Notes</u> ① In the above examples, $\vec{F}(\vec{x}) = \begin{bmatrix} y \\ -x \end{bmatrix}$ is not conservative, while $\vec{G}(\vec{x}) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ is.

② Most vector fields are not conservative. Conservative ones are special, as we will see.

<u>Def.</u> For any vector field $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$, a path $\vec{c}: \mathbb{R} \to \mathbb{R}^n$ is called a <u>flow line</u> of $\vec{F}$ if for all $t \in \mathbb{R}$, $\vec{c}\,'(t) = \vec{F}(\vec{c}(t))$

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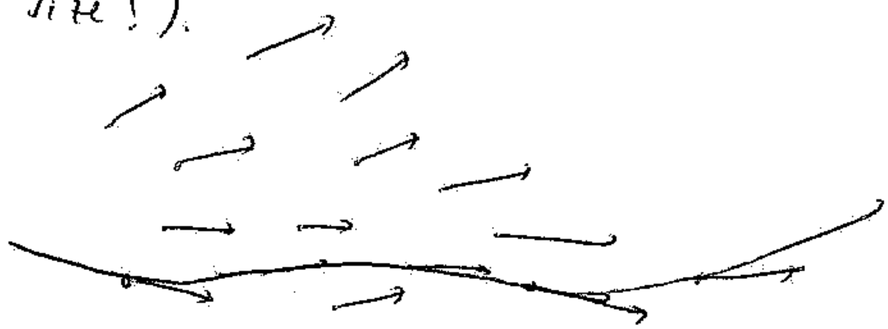
Really this means that the velocity of the path coincides with the vector field at all pts along the path (both in direction and size!).



Think of the path of a leaf in a fast moving stream with grass at the bottom of the stream.

ex. Show $\vec{c}(t) = \begin{bmatrix} e^{2t} \\ \ln|t| \\ 1/t \end{bmatrix}$ is a flow line of $F(\vec{x}) = \begin{bmatrix} 2x \\ z \\ -z^2 \end{bmatrix}$.

Soln: Need to show $\vec{c}\,'(t) = F(\vec{c}(t))$

$$\vec{c}\,'(t) = \begin{bmatrix} 2e^{2t} \\ 1/t \\ -1/t^2 \end{bmatrix} = F(\vec{c}(t)) = F(e^{2t}, \ln|t|, 1/t) = \begin{bmatrix} 2(e^{2t}) \\ 1/t \\ -1/t^2 \end{bmatrix}$$

<u>Def</u> The <u>del operator</u>, denoted ∇, is

$$\nabla = \vec{e}_1 \frac{\partial}{\partial x_1} + \cdots + \vec{e}_n \frac{\partial}{\partial x_n} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}$$

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and takes a function $f: \mathbb{R}^n \to \mathbb{R}$ to its gradient field $\nabla f: \mathbb{R}^n \to \mathbb{R}^n$.

<u>Notes</u> ① Operators are functions that take functions to functions.

ex. The differential operator D takes a function to its derivative function.

② In $\mathbb{R}^2$, $\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y}$, and in

$\mathbb{R}^3$, $\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$