Lecture 19: [redacted]

Suppose a curve $\vec{c}: [a,b] \to \mathbb{R}^n$ is not C^1 at a finite number of points. Then its length can still be computed by simply breaking up the interval at the non differentiable pts.

ex. Let $\vec{c}(t) = \begin{bmatrix} t \\ |t| \end{bmatrix}$ on $I = [-4, 4]$.

Then the arclength of $\vec{c}(t)$ is

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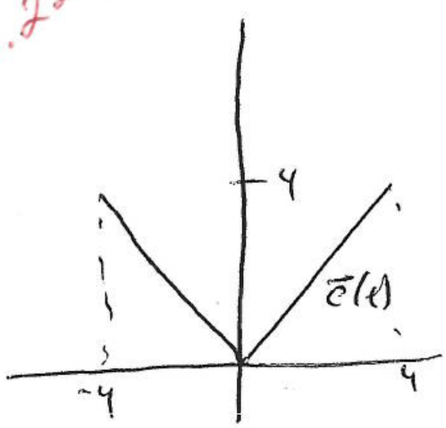
$$L(\vec{c}(t)) = \int_{-4}^{0} \|\vec{c}\,'(t)\|\, dt + \int_{0}^{4} \|\vec{c}\,'(t)\|\, dt$$



$$= \int_{-4}^{0} \sqrt{(1)^2 + (-1)^2}\, dt + \int_{0}^{4} \sqrt{(1)^2 + (1)^2}\, dt$$

Since $|t| = \begin{cases} -t & t \leq 0 \\ t & t \geq 0 \end{cases}$

$$= \int_{-4}^{0} \sqrt{2}\, dt + \int_{0}^{4} \sqrt{2}\, dt = \sqrt{2}\, t \Big|_{-4}^{0} + \sqrt{2}\, t \Big|_{0}^{4}$$

$$= 8\sqrt{2}.$$

2 more points

(I) Def A *differential* is an instantaneous change in a variable's value.

For t a variable that takes values in $\mathbb{R}$, dt is an instantaneous linear change in the value.

For 2 variables related via an equation ($y = f(x)$, for example) their differentials are also related (by $dy = f'(x)\,dx$).

Note: This looks a lot like $\frac{dy}{dx} = f'(x)$ and it is true in a sense. But $\frac{dy}{dx}$ is not a fraction! It is a notation for the derivative of the expression $y = f(x)$.

Def. if $\bar{c}(t)$ is a path, $\bar{c}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$, then displacement along $\bar{c}(t)$ is denoted by the variable s often:

Define vector displacement as

$$d\vec{s} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k} = \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} dt$$

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the infinitesimal change in ~~the coordinate~~ displacement measured along the coordinate directions. It is related to t via $\bar{c}(t)$.

Then the ~~length~~ infinitesimal change in the length of the curve $\bar{c}$ is measured by the size of $d\vec{s}$:

Define ~~the~~ the differential of displacement as

$$ds = \| d\vec{s} \| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$
$$= \sqrt{(x'(t)dt)^2 + (y'(t)dt)^2 + (z'(t)dt)^2}$$
$$= \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

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This way, we can define the length of a curve as

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$$L(\vec{c}(t)) = \int_a^b \|c'(t)\|\, dt = \int_a^b \sqrt{(x_1'(t))^2 + \cdots + (x_n'(t))^2}\, dt$$

$$= \int_a^b ds.$$

II Using this, we can define the arc length function on $\vec{c}(t)$ by

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$$s(t) = \int_a^t \|\vec{c}\,'(u)\|\, du, \quad \text{for } \vec{c} : [a,b] \to \mathbb{R}^n$$

Here $s'(t) = \|\vec{c}\,'(t)\|$ by the Fund. Thm of Calculus,

and $\displaystyle\int_a^b s'(t)\, dt = s(b) - s(a)$.

ex. Recall any function $f : [a,b] \to \mathbb{R}$ has a graph that

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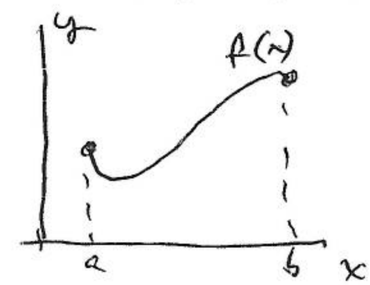
can be represented by a curve $\vec{c} : [a,b] \to \mathbb{R}^2$, $\vec{c}(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix}$



Then arclength $L(\vec{c}) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$= \int_a^b \sqrt{1 + (f'(t))^2}\, dt$ as in Calculus I.