Lecture 18: IX

Notes ① If there are more constraints, then there aren't ~~simply~~ more equations to solve for. The gradient equation simply adds a new λ for each new constraint:

The Method of Lagrange Multipliers for 2 or more constraints:

Let X be open in $\mathbb{R}^n$ and

$$f, g_1, \dots, g_k : X \subset \mathbb{R}^n \to \mathbb{R}$$

be C^1 functions, where $k<n$. Let

$$S = \{\bar{x} \in X \mid g_1(\bar{x}) = c_1, \dots, g_k(\bar{x}) = c_k\}.$$

be the intersection of the level sets of all g_i.

If $f|_S$ has an extremum $\bar{x}_0$, where $\nabla g_1(\bar{x}_0), \dots, \nabla g_k(\bar{x}_0)$ are linearly independent, then there must exist constants $\lambda_1, \dots, \lambda_k$, where

$$\nabla f(\bar{x}_0) = \lambda_1 \nabla g_1(\bar{x}_0) + \dots + \lambda_k \nabla g_k(\bar{x}_0) = \sum_{i=1}^{k} \lambda_i \nabla g_i(\bar{x}_0).$$

ex. Find all critical pts of $f(x,y,z) = 2x + y^2 - z^2$
subject to $x - 2y = 0$ and $x + z = 0$.

Solution: Here Main equation is

$$\nabla f(\vec{x}) = \lambda_1 \nabla g_1(\vec{x}) + \lambda_2 \nabla g_2(\vec{x})$$

where $g_1(x,y,z) = x - 2y = 0$, and
$g_2(x,y,z) = x + z = 0$.

We get the system:

$$\left.\begin{array}{lll} ① & x\text{-eqn:} & 2 = \lambda_1(1) + \lambda_2(1) \\ ② & y\text{-eqn:} & 2y = \lambda_1(-2) + \cancel{\lambda_2}(0) \\ ③ & z\text{-eqn:} & 2z = \lambda_1(0) + \lambda_2(1) \\ ④ & & x = 2y \\ ⑤ & & x = -z \end{array}\right\}$$

5 eqns in 5 unknowns

By ①: $\lambda_2 = 2 - \lambda_1$, ② $y = -\lambda_1$, ③ $z = \frac{\lambda_2}{2}$

and so ④ ~~&~~ and ⑤ $2y = -\cancel{z}$, or $-2\lambda_1 = \frac{\lambda_2}{2}$

By ① and this last one, $\lambda_2 = 2 - \lambda_1 = 2 + \frac{\lambda_2}{4} \Rightarrow \frac{3}{4}\lambda_2 = 2$

or $\lambda_2 = \frac{8}{3}$.

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But then $\lambda_1 = 2 - \lambda_2 = 2 - \frac{8}{3} \cancel{\lambda_2} = -\frac{2}{3}$

So that $\boxed{y = \frac{2}{3},\ z = -\frac{8}{6} = -\frac{4}{3},\ x = \frac{4}{3}}$

This is the only critical pt.

So by the Extreme Value Thm, any function $f: D \subset \mathbb{R}^n \to \mathbb{R}$ defined and continuous on a closed, bounded domain $D = U \cup \partial U$, U open, must achieve its global max and min.

How to find them?

① Find all critical pts on U via the equation
$$Df(\vec{x}) = [0, \cdots, 0].$$

② Find all constrained critical pts on ∂U via Method of Lagrange Multipliers.

③ Evaluate f on all critical pts of ① & ② and choose largest and smallest.

Section 4.1 Back to paths as our first examples of vector-valued functions.

Here there are only a few things to add:

(I) The derivative of a path $\vec{c}: \mathbb{R} \to \mathbb{R}^n$
$\vec{c}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, is another path $\vec{c}\,'(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$

which may also be differentiable. Here $\vec{c}\,''(t)$, if defined is called the acceleration of $\vec{c}(t)$. And so on...

(II) Products (dot, cross in $\mathbb{R}^3$), sum, difference, constant multiples all behave normally for derivatives.

(III) Recall, a C^1 path can have corners in its image (see cycloid). This is ~~bec~~ true when $\vec{c}\,'(t) = 0$ at the corner.

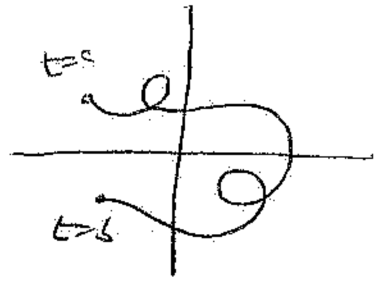
Def A C^1 path is called regular if $\vec{c}\,'(t) \neq 0$ for all t in the domain of $\vec{c}(t)$!

ex. $\vec{c}(t) = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix}$ is regular on $(0, 2\pi)$ but not on $[-\pi, \pi]$.

Only pg 217–219 are particularly important for us.

Section 4.2 A particularly important aspect of a path is its length on an interval:

$\vec{c}: [a,b] \longrightarrow \mathbb{R}^2$



How far did the bead on this wire travel?

How long is this path?

If $\vec{c}$ is differentiable, then the velocity is $\vec{c}\,'(t)$ and the speed at time t is $\|\vec{c}\,'(t)\|$ (the size of the vector velocity).

Speed is distance traveled over an interval of time.

We integrate speed to recover distance (like in calc I).

$$L(\vec{c}(t)) = \int_a^b \|\vec{c}\,'(t)\|\, dt$$

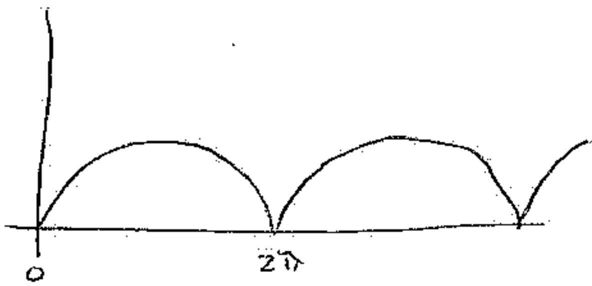
Comes from a Riemann Sum or small straight lines approximating the curve.

$$= \int_a^b \sqrt{(x_1'(t))^2 + \cdots + (x_n'(t))^2}\, dt$$

for $\vec{c}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$.

ex. ~~How~~ What is the length of one arch in the cycloid:

$$\vec{c}(t) = \begin{bmatrix} t-\sin t \\ 1-\cos t \end{bmatrix}, \quad \vec{c}: \mathbb{R} \to \mathbb{R}^2$$



Solution: $\vec{c}\,'(t) = \begin{bmatrix} 1-\cos t \\ -\sin t \end{bmatrix}$.

So

$$L(\text{one arch of cycloid}) = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (-\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\left(\tfrac{t}{2}\right)}\, dt, \text{ since } \sin^2\tfrac{t}{2} = \tfrac{1}{2} - \tfrac{1}{2}\cos t \text{ (remember this?)}$$

and on $[0, 2\pi]$, $\sin\frac{t}{2} \geq 0$.

$$= \int_0^{2\pi} 2\sin\left(\tfrac{t}{2}\right) dt = -4\cos\tfrac{t}{2}\Big|_0^{2\pi} = 8.$$

Note: If a curve (path) is not C^1, ~~[illegible]~~ at a finite number of places (bounces off walls, for instance), then we can still integrate to find its length. Just break up the integral at the places where $\vec{c}$ is not C^1.