Lecture 16: [redacted]

<u>Def.</u> Let $f:\mathbb{R}^n \to \mathbb{R}$ be C^2 at $\vec{x}_0$. Then the Hessian Matrix of f at $\vec{x}_0$ is

$$Hf(\vec{x}_0) = \underbrace{\left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]}_{n} \Big\} n$$

the matrix of 2nd partials of f eval. at $\vec{x}_0$.

<u>Notes</u> ① The book defines $Hf(\vec{x}_0)$ as a function on $\mathbb{R}^n$:

$$Hf(\vec{x}_0)(\vec{h}) = \frac{1}{2} \sum_{i,j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) h_i h_j$$

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$$= \frac{1}{2} \vec{h}^T \left[\begin{matrix} \frac{\partial^2 f}{\partial x_i \partial x_j} \\ \text{matrix} \end{matrix} \right] \vec{h}$$

It is precisely the entire 2nd derivative term of $T_2(f, \vec{x}_0)$, defined previously

② It is much more normal to call just the matrix $Hf(\vec{x}_0)$. I will always call the matrix of 2nd partials the Hessian <u>matrix</u>, to be clear.

Notes on the Hessian Matrix (cont'd)

③ Hessian matrices are always symmetric: A matrix

$$\underset{n\times n}{A} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{n} \Big\} n$$ is symmetric if for all $i, j = 1, \ldots, n$, $a_{ij} = a_{ji}$.

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(mixed partials are equal).

④ The Hessian function $Hf(\vec{x}_0): \mathbb{R}^n \to \mathbb{R}$,

$$Hf(\vec{x}_0)(\vec{h}) = \frac{1}{2}\vec{h}^T\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]\vec{h}$$ is

- positive definite if for all $\vec{h} \in \mathbb{R}^n$,

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$Hf(\vec{x}_0)(\vec{h}) \geq 0$ and $Hf(\vec{x}_0)(\vec{h}) = 0$ only when $\vec{h} = \vec{0}$.

- negative definite if for all $\vec{h} \in \mathbb{R}^n$

$Hf(\vec{x}_0)(\vec{h}) \leq 0$ and $Hf(\vec{x}_0)(\vec{h}) = 0$ only when $\vec{h} = \vec{0}$.

⑤ Positive definite and negative definite can also be seen directly in the Hessian Matrix also:

Notes on the Hessian Matrix (cont'd.)

⑤ cont'd. A symmetric $n \times n$ matrix $A_{n \times n}$ is positive definite if ⓐ its determinant is positive, and ⓑ all of its diagonal sub-determinants are positive

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$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

A is pos. def if $a_{11} > 0$, and $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0$, and $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} > 0$, and ... ... and $\det A > 0$.

A symmetric matrix is called negative definite if the determinants alternate between positive and negative starting with $a_{11} < 0$.

~~The other possibility~~

Now we can use this information to create a multivariable 2nd derivative test for the possible classification of a critical pt.

<u>Thm</u> if $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is C^2 at the critical pt $\bar{x}_0 \in U$, and $Hf(\bar{x}_0)$ is <u>positive</u> <u>definite</u>, then $\bar{x}_0$ is a <u>local</u> <u>minimum</u> for f. if $Hf(\bar{x}_0)$ is negative definite, then $\bar{x}_0$ is a local maximum.

Here is a special case for $n=2$:

<u>Thm</u> Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ be C^2 on an open U, and $\bar{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in U$ be critical for f. Then if

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(upper left entry) (i) $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$, and

(det. of $Hf(\vec{x})$) (ii) $\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ at (x_0, y_0),

$\vec{x}$ is a local min.

det $Hf(\vec{x}) \neq 0$ a pt is nondegenerate. o/w is degenerate.

if (ii) is the same, but (i) is negative, then $\bar{x}_0$ is a local maximum.

if (ii) is negative, then $\vec{x}_0$ is a saddle and not an extremum.

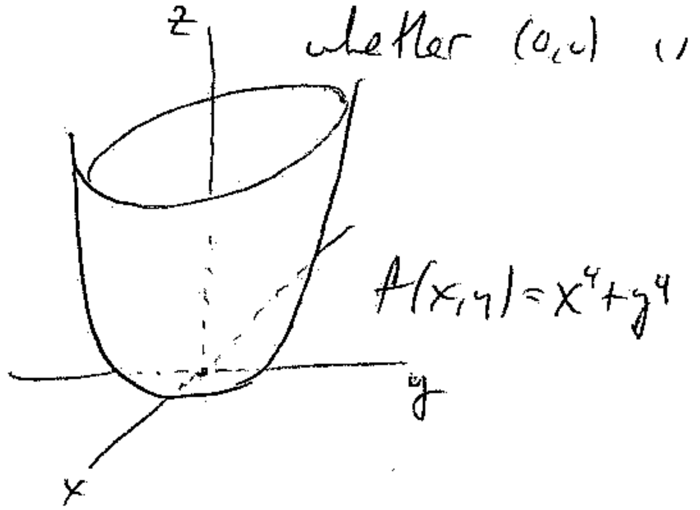
All other cases we just cannot determine w/ $Hf(\bar{x}_0)$.

<u>example</u> $f(x,y) = x^4 + y^4$

Here $(x,y) = (0,0)$ is critical, since $Df(0,0) = [0 \ 0]$,

But
$$Hf(0,0) = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here the 2nd derivative Test fails to determine whether $(0,0)$ is extreme or not. (it is a min).



This graph looks like x^2+y^2 but the parabolic bowl got heavy and flattened out at the bottom.

<u>example</u> Compare this with $g(x,y) = -x^4 - y^4 = -f(x,y)$

Here $(0,0)$ is a local max, but again

$$Df(0,0) = [0 \ 0], \quad Hf(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the 2nd derivative test is not helpful.

<u>Note</u>: This was true in calc I given the functions $f(x) = x^4$ and $g(x) = -x^4$.

example $f(x,y) = x^2 - y^2$ (the saddle).

Here $(x,y) = (0,0)$ is critical since

$$Df(0,0) = [2x, -2y]\Big|_{(0,0)} = [0 \ \ 0].$$

And $Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$. Here in the Theorem,

(i) was >0 but (ii) was <0. This pt is a saddle and not extreme.

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example $g(x,y) = x^2 + y^2$

Here $(0,0)$ is critical, since $Df(0,0) = [2x \ \ 2y]\Big|_{(0,0)} = [0 \ \ 0]$

And $Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, is positive definite.

So by 2nd Derivative test, $(0,0)$ is a local min.

Here (i) $\frac{\partial^2 f}{\partial x^2}(0,0) = 2 > 0$, and

(ii) $\left(\frac{\partial^2 f}{\partial x^2}(0,0)\right)\left(\frac{\partial^2 f}{\partial y^2}(0,0)\right) - \left(\frac{\partial^2 f}{\partial x \partial y}(0,0)\right)^2 = 2(2) - 0 = 4 > 0$

Final thoughts:

Def. For $f: U \subset \mathbb{R}^n \to \mathbb{R}$, a pt $\vec{x}_0 \in U$ is called a global min (max) if for all $\vec{x} \in U$, $f(\vec{x}) \geq f(\vec{x}_0)$ $(f(\vec{x}) \leq f(\vec{x}_0))$.

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Def. A set $D \subset \mathbb{R}^n$ is said to be bounded if there is a number $M \in \mathbb{R}$ such that for all $\vec{x} \in D$, $\|\vec{x}\| < M$.

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The Extreme Value Thm

For $U \subset \mathbb{R}^n$ closed and bounded, if $f: U \to \mathbb{R}$ is continuous, then f has a global minimum and global maximum on U.

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Note: Do you recognize this thm when $n=1$ from SVC?