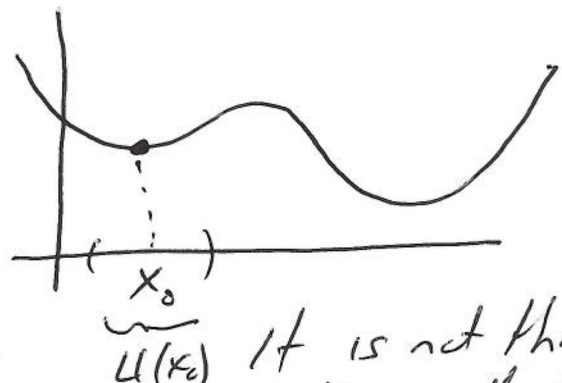
Lecture 15: [redacted]

Recall from Single Variable Calculus:

$\mathbb{R}^n$-version p. 168

<u>Def</u> A pt x_0 in the domain of $f: \mathbb{R} \to \mathbb{R}$ is a <u>local</u> <u>minimum</u> if there is a neighborhood of x_0, $U(x_0) \subset \mathbb{R}$, where for all $x \in U(x_0)$, $f(x) \geq f(x_0)$.



(Literally, among its nearest neighbors, x_0 has the lowest function value)

It is not that x_0 is the smallest value for f on every open set containing x_0. Just one.

<u>Notes</u> ① There is a similar definition for local maximum with the inequality reversed.

② Recall that a neighborhood in $\mathbb{R}$ means that one must check both sides of x_0.

③ A local minimum or a local maximum is also called an <u>extremum</u>.

For functions $f: \mathbb{R}^n \to \mathbb{R}$, the definition is the same, although the idea of a neighborhood is different (must include nearby points in all directions

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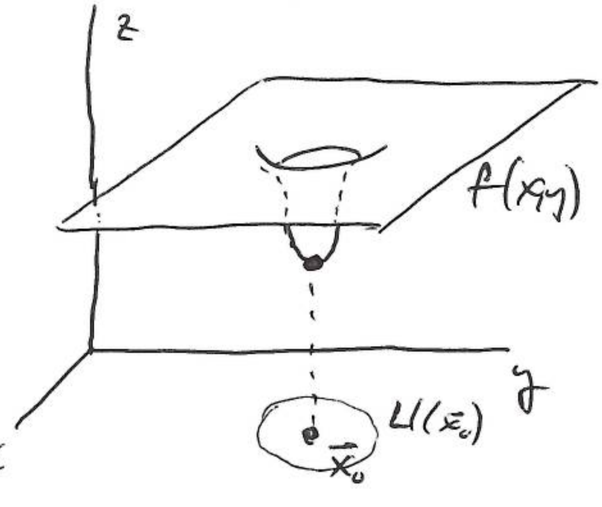
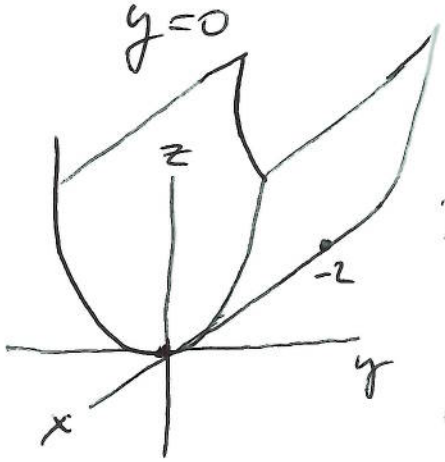
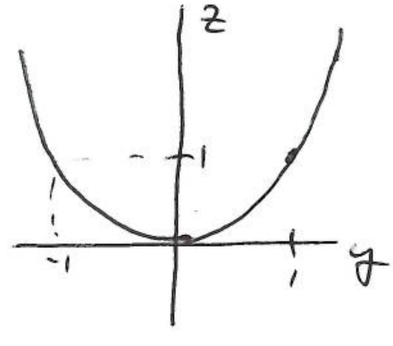


Do keep in mind that local extrema are not always isolated points.

ex. $z = g(y) = y^2$ has an isolated local minimum at $y=0$





But the function $z = f(x,y) = y^2$ defined on the domain

$U = \{(x,y) \in \mathbb{R}^2 \mid -2 \le x \le 0\}$ has a line of local minima along the negative x-axis.

Def. Given $f: U \subset \mathbb{R}^n \to \mathbb{R}$, a pt $\bar{x}_0 \in U$ is called <u>critical</u> for f if either

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i) $Df(\bar{x}_0)$ is not defined, or

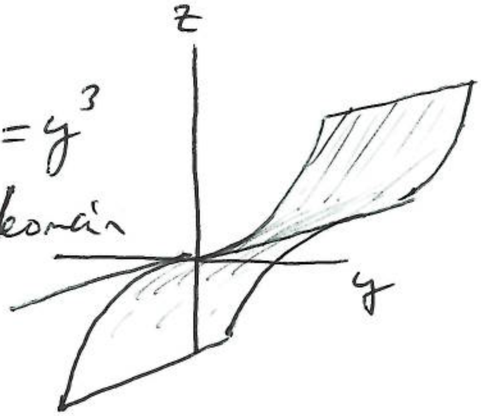
ii) $Df(\bar{x}_0) = [0 \cdots 0]$ (is the 0-matrix).

<u>Notes</u> ① Compare with critical pt definition in Calc I.

② Can an ~~and~~ critical pt be neither a local max or a local min?

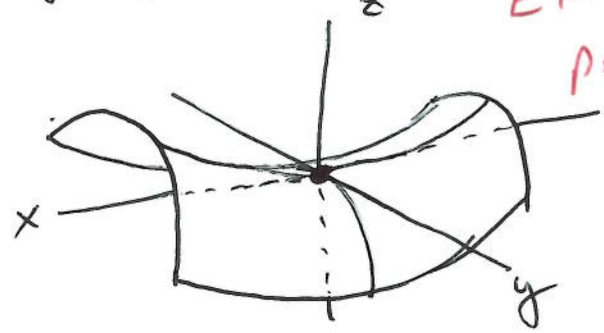
$f(x,y) = y^3$

on same domain as last <u>example</u>.



$g(x,y) = x^2 - y^2$

Ex. 2 p. 170



At left, there is a line of nonextreme critical pts along the negative x-axis.

At right is an isolated non extreme critical pt for $g(x,y)$. Can you see why it is neither a local max nor a local min?

③ Critical pts are simply points where possibly interesting function behavior occurs. (a function stops rising and starts falling, for example).

④ So critical pts need not be extreme, But extreme pts, if they occur, are always critical!

<u>Thm</u> if $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{x}_0 \in U$ (U is an open set) and $\vec{x}_0$ is extreme, then $\vec{x}$ is also critical, and $Df(\vec{x}_0) = [0, \cdots, 0]$.

Thm 4 p. 169

example 4, pg 171. Find the critical pts of

$$f(x,y) = 2(x^2+y^2)e^{-(x^2+y^2)}.$$

<u>Solution</u>: ~~the~~ ~~a~~ Since f is C^1 on all of $\mathbb{R}^2$, we just need to calculate where $Df(\vec{x}) = [0, 0]$.

Here
$$\frac{\partial f}{\partial x}(\vec{x}) = 2(2x)e^{-(x^2+y^2)} + 2(x^2+y^2)e^{-(x^2+y^2)}(-2x)$$
$$= 4xe^{-(x^2+y^2)}(1-x^2-y^2)$$

And $\frac{\partial f}{\partial x}(\vec{x}) = 0$ when $x=0$ or along the circle $x^2+y^2=1$.

ex (cont'd)

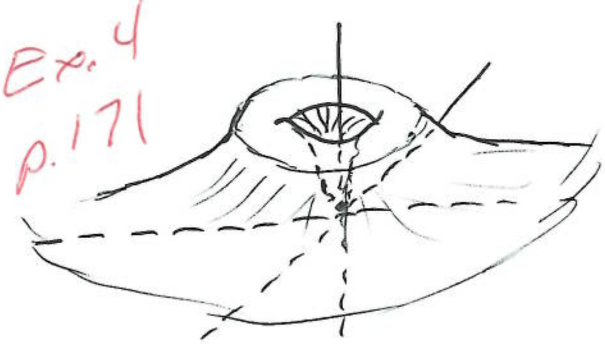
And similarly, $\frac{\partial f}{\partial y}(\vec{x}) = 4y\, e^{-(x^2+y^2)}(1-x^2-y^2)$

and $\frac{\partial f}{\partial y}(\vec{x}) = 0$ when $y=0$ and along $x^2+y^2=1$.

So $Df(\vec{x}) = [0 \;\; 0]$ where both are simultaneously true:

at $(0,0)$ and along the domain circle $1 = x^2+y^2$

See drawing in book. Here Ex. 4 p. 171



$f(x,y)$ is called the volcano function, and looks like one: The hole in the middle bottoms out at the origin $(0,0)$, the local min, and the doughnut like peak is along the circle $1 = x^2+y^2$.

Q: Classifying (possibly) a critical pt as extreme in Calculus I sometimes involved the 2^{nd} derivative test. Is there such a thing in vector Calculus?

A: Yes, although here we have a matrix $\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\vec{x}) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\vec{x}) \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\vec{x}) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(\vec{x}) \end{bmatrix}$