Lecture 13: [redacted] $\underline{\text{V}}$

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Thm Let $f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be C^1 at $\vec{x}_0 \in \mathbb{R}^n$, and for $f(\vec{x}_0) = k_0 \in \mathbb{R}$, define

$$S_{k_0} = \{ \vec{x} \in \mathbb{R}^n \mid f(\vec{x}) = k_0 \}$$

as the k_0-level set of f containing $\vec{x}_0 \in \mathbb{R}^n$.

Then we have:

1. $\nabla f(\vec{x}_0)$ is normal to S_{k_0} in the following sense: Let $\vec{c}(t)$ be a differentiable curve in S_{k_0}, with $\vec{c}(0) = \vec{x}_0$.

   Then for $\vec{v} = \vec{c}\,'(0)$, $\nabla f(\vec{x}_0) \cdot \vec{v} = 0$.

2. As long as $\nabla f(\vec{x}_0) \neq \vec{0}$, the tangent space to S_{k_0} at $\vec{x}_0$ is given by the equation

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0 .$$

Q: Do you recognize this last equation?

Notes written out:

① In $\mathbb{R}^2$, with $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, this last equation is

$$0 = \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) \\ \frac{\partial f}{\partial y}(x_0, y_0) \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$$0 = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)(y - y_0)$$

② In $\mathbb{R}^3$, with $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, we get

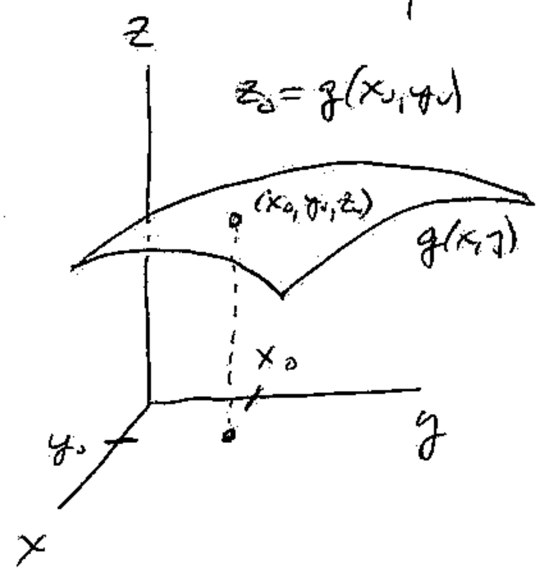
$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0, z_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0, z_0)\right)(y - y_0) + \left(\frac{\partial f}{\partial z}(x_0, y_0, z_0)\right)(z - z_0) = 0.$$

Do you see the pattern?

Compare the equation for the tangent space to a level set of $f(x,y,z)$ as a subset of $\mathbb{R}^3$, $\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$, to the equation of the tangent space to the graph of $z = g(x,y)$ as a subset of $\mathbb{R}^3$.

They are precisely the same, when $f(x,y,z) = g(x,y) - z$:

ex. Given any function $g: \mathbb{R}^2 \to \mathbb{R}$, its graph is $z = g(x,y)$ and will look like a surface in xyz-space $\mathbb{R}^3$.



If g is differentiable at (x_0, y_0) then the tangent space exists to graph(g) and is given by

$$z = g(x_0,y_0) + \left(\frac{\partial g}{\partial x}(x_0,y_0)\right)(x-x_0) + \left(\frac{\partial g}{\partial y}(x_0,y_0)\right)(y-y_0)$$

(See pg 110 here). Call this (eqn 1)

Now create a function $f: \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = g(x,y) - z$. We cannot see the graph(f) since it "lives" in $\mathbb{R}^4$. But we can study its level sets, since they will look like surfaces in $\mathbb{R}^3$.

In fact, the 0-level set of f: $S_0 = \{\vec{x} \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$ satisfies $0 = f(x,y,z) = g(x,y) - z$.

And graph(g) is also the set of all points $(x,y,z) \in \mathbb{R}^3$ where $g(x,y) = z$.

Here graph(g) = (the 0-level set of f).

So what is the equation for the tangent plane to f at (x_0, y_0, z_0): For $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, it is

$$\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0 = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0,z_0) \\ \frac{\partial f}{\partial y}(x_0,y_0,z_0) \\ \frac{\partial f}{\partial z}(x_0,y_0,z_0) \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}.$$

$$0 = \left(\frac{\partial f}{\partial x}(x_0,y_0,z_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0,y_0,z_0)\right)(y - y_0) + \left(\frac{\partial f}{\partial z}(x_0,y_0,z_0)\right)(z - z_0)$$

Call this (eqn 2).

Q: Are (eqn 1) and (eqn 2) the same?

Hint: They define the same tangent plane, so they should be.

A: lets find out:

ⓐ $f(x_0, y_0, z_0) = g(x_0, y_0) - z_0 = 0$

since $z_0 = g(x_0, y_0)$. Hence $\boxed{z - z_0 = z - g(x_0, y_0)}$

ⓑ $\frac{\partial f}{\partial z}(x_0, y_0, z_0) = \frac{\partial}{\partial z}\left(g(x,y) - z\right)\Big|_{(x_0, y_0, z_0)} = -1$

ⓒ $\frac{\partial f}{\partial x}(x_0, y_0, z_0) = \frac{\partial}{\partial x}\left(g(x,y) - z\right)\Big|_{(x_0, y_0, z_0)} = \frac{\partial g}{\partial x}(x_0, y_0)$ (why?)

ⓓ $\frac{\partial f}{\partial y}(x_0, y_0, z_0) = \frac{\partial}{\partial y}\left(g(x,y) - z\right)\Big|_{(x_0, y_0, z_0)} = \frac{\partial g}{\partial y}(x_0, y_0)$

Hence (eqn 2) is

$$0 = \left(\frac{\partial f}{\partial x}(x_0, y_0, z_0)\right)(x - x_0) + \left(\frac{\partial f}{\partial y}(x_0, y_0, z_0)\right)(y - y_0) + \left(\frac{\partial f}{\partial z}(x_0, y_0, z_0)\right)(z - z_0)$$

$$= \left(\frac{\partial g}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial g}{\partial y}(x_0, y_0)\right)(y - y_0) + (-1)\left(z - g(x_0, y_0)\right)$$

or $$z = g(x_0, y_0) + \left(\frac{\partial g}{\partial x}(x_0, y_0)\right)(x - x_0) + \left(\frac{\partial g}{\partial y}(x_0, y_0)\right)(y - y_0)$$

which is (eqn 1).

They are the same ~~when~~ when the graph(g) is the 0-level set of $f(x,y,z) = g(x,y) - z$.

If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is C^1, then all partials exist and are continuous functions. They are all real-valued functions on U also: For each $i = 1, \ldots, n$, $\frac{\partial f}{\partial x_i}: U \subset \mathbb{R}^n \to \mathbb{R}$

If each of these partial derivative functions is also differentiable, or C^1, then each has n-partials of its own.

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In this case, we call the original a C^2 function, with continuous partials up to order 2.

Note: C^2 means order-2 continuous if one considers differentiability a kind of continuity.

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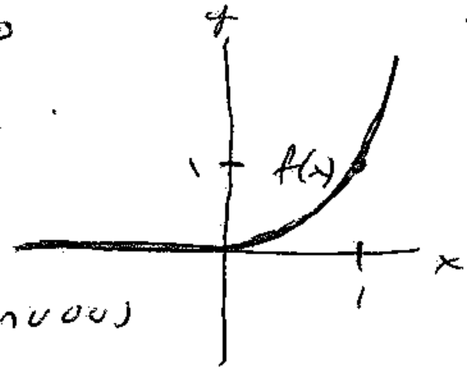
Any function that is C^n continuous is also C^m continuous for each $0 \leq m < n$.

But there are always functions that are C^n continuous but not C^{n+1} continuous.

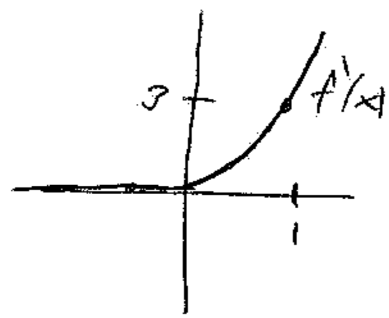
ex. Let $f(x) = \begin{cases} x^3 & x>0 \\ 0 & x \leq 0 \end{cases}$



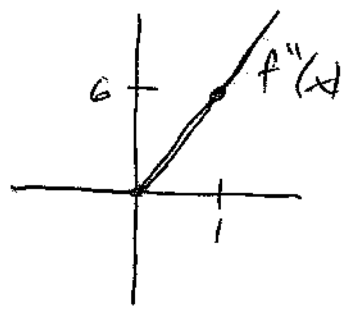
- Here f is certainly continuous so $f \in C^0$. (f is an element of the set of C^0 functions on $\mathbb{R}$)
- f is differentiable, since

$$f'(x) = \begin{cases} 3x^2 & x>0 \\ 0 & x \leq 0 \end{cases}$$

is continuous ($f \in C^1$)



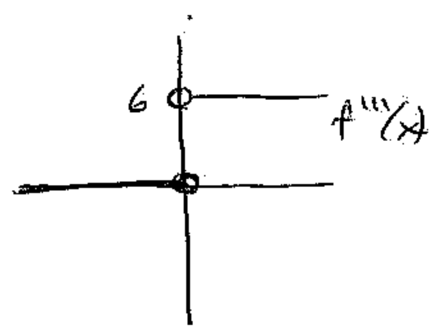
- f is C^2 since $f''(x) = \begin{cases} 6x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$ is continuous. ($f \in C^2$)



- But f is not C^3, since

$$f'''(x) = \begin{cases} 6 & x>0 \\ 0 & x<0 \end{cases}$$

is not continuous at $x=0$.



ex. Let $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xy^2 + y^2z^3 + z^3x$.

Then f is C^1, since

$$\frac{\partial f}{\partial x}(x,y,z) = y^2 + z^3, \quad \frac{\partial f}{\partial y}(x,y,z) = 2xy + 2yz^3, \quad \frac{\partial f}{\partial z}(x,y,z) = 3y^2z^2 + 3xz^2$$

are all continuous functions (they are all polynomials)

But since they are all polynomials, they themselves are differentiable, making f a C^2 function, and

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(y^2 + z^3\right) = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(y^2 + z^3\right) = 2y$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial z}\left(y^2 + z^3\right) = 3z^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2xy + 2yz^3\right) = 2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(2xy + 2yz^3\right) = 2x + 2z^3$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}\left(\quad\downarrow\quad\right) = 6yz^2$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial x}\left(3y^2z^2 + 3xz^2\right) = 3z^2$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial y}\left(\quad\downarrow\quad\right) = 6yz^2$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(\quad\downarrow\quad\right) = 6y^2z + 6xz$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z}$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}$$

Is it possible this is always true??

Do you see any patterns??

A: Yes, as long as the functions are "nice".

<u>Thm</u> If $f: \mathbb{R}^2 \to \mathbb{R}$ is C^2 (2nd partials exist and are continuous functions in a nbhd of each pt) then the <u>mixed partials</u> are equal, so that the order of ~~integration~~ differentiation does not matter.

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This means $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

<u>Thm</u> For any C^2 function $f: \mathbb{R}^n \to \mathbb{R}$ the order of differentiation in all mixed partials does not matter.

ex. Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is C^4 (for example). Then it is also C^2, and

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}, \quad \cdots, \text{ and.}$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{\partial^3 f}{\partial z \partial y \partial x} = \frac{\partial^3 f}{\partial y \partial x \partial z} = \cdots \quad \text{(for all 6 possibilities)}$$