Lecture 11: [redacted]

Given $f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ a C^1-real valued function on $\mathbb{R}^n$, recall that $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$

is an n-vector of functions. This means that for $f: U \subset \mathbb{R}^n \to \mathbb{R}$, $\nabla f: U \subset \mathbb{R}^n \to \mathbb{R}^n$, where ∇f still has n-inputs, but also now has n-outputs (the partials), each of which is a real-valued function on $U \subset \mathbb{R}^n$:

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$$\nabla f(x_1, \cdots, x_n) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x_1, \cdots, x_n) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_1, \cdots, x_n) \end{bmatrix}$$

and only when evaluated at a pt $\vec{x}_0 \in U \subset \mathbb{R}^n$,

$$\nabla f(\vec{x}_0) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(\vec{x}_0) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}_0) \end{bmatrix}$$ is an n-vector in $\mathbb{R}^n$.

<u>ex</u>. For $f(x,y,z) = xy^2 + y^2z^3 + z^3x$,

find $\nabla f(2,-1,1)$.

<u>Solution</u>: Here $\nabla f(x,y,z) = \begin{bmatrix} y^2+z^3 \\ 2xy+2yz^3 \\ 3y^2z^2+3xz^2 \end{bmatrix}$

so that $\nabla f(2,-1,1) = \begin{bmatrix} (-1)^2+(1)^2 \\ 2(2)(-1)+2(-1)(1)^3 \\ 3(-1)^2(1)^2+3(2)(1)^2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 9 \end{bmatrix}$.

Note that $\nabla f(2,-1,1)$ is a vector in $\mathbb{R}^3$ based at $(2,-1,1)$ and <u>not</u> at the origin.

Here $\nabla f(\vec{x}_0)$ is actually an example of a "<u>directional</u> <u>derivative</u>": A measure of how a function $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is changing along a particular direction in the domain.

Compare this to the examples of the Chain Rule above involving a parameterized curve.

Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is C^1-real valued, so at any pt $\vec{x}_0 \in \mathbb{R}^n$, all partials exist and are continuous functions "around" $\vec{x}_0$.

Then the <u>directional</u> <u>derivative</u> of f at $\vec{x}_0 \in \mathbb{R}^n$ in the direction $\vec{v} \in \mathbb{R}^n$ is

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$$\frac{d}{dt} f(\vec{x}_0 + t\vec{v})\Big|_{t=0}$$

<u>Notes</u> ① This is a composition of 2 functions here: ⓐ the outside function is $f: \mathbb{R}^n \to \mathbb{R}$ and ⓑ the inside function is a param. curve $\vec{c}: \mathbb{R} \to \mathbb{R}^n$, $\vec{c}(t) = \vec{x}_0 + t\vec{v}$. Hence $f(\vec{x}_0 + t\vec{v}): \mathbb{R} \to \mathbb{R}$ and the notation $\frac{d}{dt}(\cdot)$ indicates a SVC derivative (see examples in last lecture).

Notes (cont'd.)

② The parameterized line $\vec{x}_0 + t\vec{v}$ above indicates that at $t=0$, is the pt $\vec{x}_0$ and at $t=1$, one has traversed the length of $\vec{v}$ on the line. This means that for different length choices of $\vec{v}$ you will see different values of the directional derivatives (see example below)

In order to ensure the derivative is an accurate measurement of how f is changing, $\vec{v}$ must be chosen of unit length (to be compatible with the notions of length along the $x_1, \ldots, x_n$ directions in $\mathbb{R}^n$).

$$\boxed{\text{Only choose } \vec{v} \in \mathbb{R}^n, \text{ where } \|\vec{v}\| = 1.}$$

Notes cont'd

(3) In practice, for $f: \mathbb{R}^n \to \mathbb{R}$ C^1,

$$\frac{d}{dt} f(\vec{x}_0 + t\vec{v})\Big|_{t=0} = Df\left((\vec{x}_0 + t\vec{v})\Big|_{t=0}\right) \cdot \frac{d}{dt}\Big|_{t=0} (\vec{x}_0 + t\vec{v})$$

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$$= Df(\vec{x}_0) \underset{\substack{\uparrow \\ \text{matrix} \\ \text{product}}}{\cdot} \vec{v} = \nabla f(\vec{x}_0) \underset{\substack{\uparrow \\ \text{dot} \\ \text{product}}}{\cdot} \vec{v}$$

$$= \left(\frac{\partial f}{\partial x_1}(\vec{x}_0)\right) v_1 + \left(\frac{\partial f}{\partial x_2}(\vec{x}_0)\right) v_2 + \cdots + \left(\frac{\partial f}{\partial x_n}(\vec{x}_0)\right) v_n$$

where $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\|\vec{v}\| = 1$.

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ex Incorrect and correct calculation.

Let $f: \mathbb{R}^3 \to \mathbb{R}$ be as in the above example, $f(x,y,z) = xy^2 + y^2z^3 + z^3x$. Calculate the directional derivative of f in the direction of $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ at $\vec{x}_0 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Solution: We already know $\nabla f(\vec{x}_0) = \nabla f(2,-1,1) = \begin{bmatrix} 2 \\ -6 \\ 9 \end{bmatrix}$. Hence

$$\frac{d}{dt} f\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right)\Bigg|_{t=0} = \nabla f(2,-1,1) \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -6 \\ 9 \end{bmatrix} \underset{\text{dot prod.}}{\cdot} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 2(1) - 6(2) + 9(2) = 8$$

It looks like f is increasing rapidly in this direction at $\vec{x}_0$.

However, this is not correct! Here

$$\|\vec{v}\| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3, \text{ for } \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

To accurately calculate the directional derivative, first find the unit vector in the direction of $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. It is $\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \vec{w}$.

Then $\frac{d}{dt} f(\vec{x}_0 + t\vec{w})\Big|_{t=0} = \begin{bmatrix} 2 \\ -6 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 1/3 \\ 2/3 \\ 4/3 \end{bmatrix} = \frac{2}{3} - \frac{12}{3} + \frac{18}{3} = \frac{8}{3}$.