Lecture 10: [redacted]

<u>Special Note:</u> The book mentions differentiability and C^1 as two different things. It is correct, but do not worry about this.

Think that differentiability as: All partials exist and are continuous in a nbhd of the point.

For $\vec{f}: U \subset \mathbb{R}^n \to \mathbb{R}^m$ differentiable (as above), all general derivative rules from SVC also apply to $\vec{f}$, and $D\vec{f}(\vec{x}_0)$, as long as the ~~Ru~~ Rules make sense!

Here this means as long as the requisite parts are compatible.

(I) Constant Multiple Rule: Let $\vec{h}: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be defined as $\overrightarrow{h(\vec{x})} = c\overrightarrow{f(\vec{x})}$ for $c \in \mathbb{R}$. Then

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$$D\overrightarrow{h(\vec{x})} = D(c\vec{f})(\vec{x}) = c\, D\overrightarrow{f(\vec{x})}$$

since any size matrix can be multiplied by $c \in \mathbb{R}$.

II Sum Rule: Let $\vec{f}: U \subset \mathbb{R}^n \to \mathbb{R}^m$
$\vec{g}: V \subset \mathbb{R}^p \to \mathbb{R}^q$. Create

$\overrightarrow{h(\vec{x})} = \overrightarrow{f(\vec{x})} + \overrightarrow{g(\vec{x})}$. Then this only makes sense if $p=n$ and $U=V$. And I can only add vectors $\overrightarrow{f(\vec{x})}$ and $\overrightarrow{g(\vec{x})}$ if they are the same size (so $m=q$). So let

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$$\left.\begin{array}{l}\vec{f}: U \subset \mathbb{R}^n \to \mathbb{R}^m \\ \vec{g}: U \subset \mathbb{R}^n \to \mathbb{R}^m\end{array}\right\} \text{ if } \overrightarrow{h(\vec{x})} = \overrightarrow{f(\vec{x})} + \overrightarrow{g(\vec{x})},$$

Then $$D\overrightarrow{h(\vec{x})} = D(\vec{f}+\vec{g})(\vec{x}) = D\overrightarrow{f(\vec{x})} + D\overrightarrow{g(\vec{x})}$$

and all matrices are $m \times n$.

III Product Rule: If $\vec{f}, \vec{g}$ are like in the beginning of II above, and we try to create $\overrightarrow{h(\vec{x})} = \overrightarrow{f(\vec{x})}\,\overrightarrow{g(\vec{x})}$, the only way this is possible is for both $\vec{f}, \vec{g}$ have $U \subset \mathbb{R}^n$ as their domain and for $m=q=1$ (otherwise we cannot multiply $\overrightarrow{f(\vec{x})}$ and $\overrightarrow{g(\vec{x})}$

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III cont'd.

So let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ be real-valued

Then the Product Rule holds, and for $h(\vec{x}) = f(\vec{x})g(\vec{x})$ (no vector-valued), we have

$$\underbrace{Dh(\vec{x})}_{\substack{1\times n \\ \text{matrix}}} = \underbrace{g(\vec{x})}_{\substack{\text{Real} \\ \#}} \underbrace{Df(\vec{x})}_{\substack{1\times n \\ \text{matrix}}} + \underbrace{f(\vec{x})}_{\substack{\text{real} \\ \#}} \underbrace{Dg(\vec{x})}_{\substack{1\times n \\ \text{matrix}}}$$

Same type of thing for the Quotient Rule.

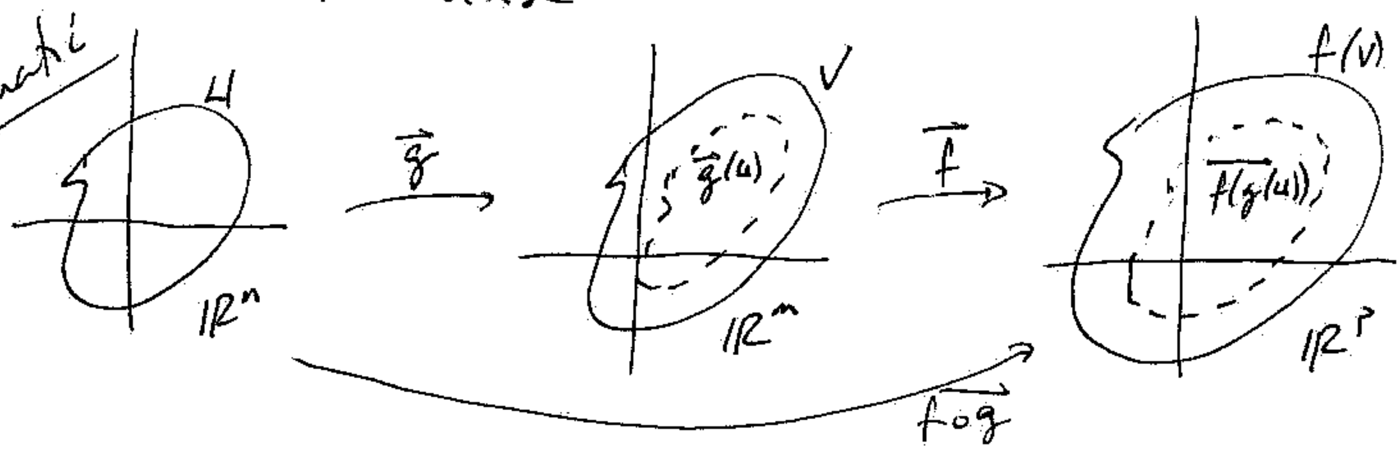
IV Chain Rule

Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets, and

$\vec{g} : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $\vec{f} : V \subset \mathbb{R}^m \to \mathbb{R}^p$ be functions,

where $\vec{g}(U) \subset V$ (so that $(\vec{f} \circ \vec{g}) : U \subset \mathbb{R}^n \to \mathbb{R}^p$ makes sense

Schematic



Now suppose that $\vec{g}$ is differentiable at $\vec{x}_0$ and $\vec{f}$ is differentiable at $\vec{g}(\vec{x}_0)$. Then $\overrightarrow{f \circ g}$ is differentiable at $\vec{x}_0$, and

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$$\underbrace{D(\overrightarrow{f \circ g})(\vec{x}_0)}_{\substack{p \times n \\ \text{matrix}}} = \underbrace{D\vec{f}(\vec{g}(\vec{x}_0))}_{\substack{p \times m \\ \text{matrix}}} \underset{\substack{\uparrow \\ \text{matrix} \\ \text{mult.}}}{\cdot} \underbrace{D\vec{g}(\vec{x}_0)}_{\substack{m \times n \\ \text{matrix}}}$$

How does this work in practice?

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ex. Let $\vec{c}: \mathbb{R} \to \mathbb{R}^3$ be a diff. curve in $\mathbb{R}^3$, and $f: \mathbb{R}^3 \to \mathbb{R}$ be a C^1-real-valued function.

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Q: How is f changing along the curve $\vec{c}$?

Note: In general, this question may not make sense, but since $(f \circ \vec{c}): \mathbb{R} \to \mathbb{R}$, like in SVC, it does:

A: Calculate the derivative of $(f \circ \vec{c}): \mathbb{R} \to \mathbb{R}$, where

$(f \circ \vec{c})(t) = f(\vec{c}(t)) = f\left(\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}\right) \in \mathbb{R}$, for a choice of $t \in \mathbb{R}$, since $\vec{c}: \mathbb{R} \to \mathbb{R}^3$, $\vec{c}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$.

Since both f and $\vec{c}$ are differentiable everywhere, by the Chain Rule

$$\underbrace{\frac{d}{dt}(f\circ\vec{c})(t_0) = D(f\circ\vec{c})(t_0)}_{\substack{\text{a real number}\\ \text{(same as a}\\ \text{1×1 matrix)}}} = \underbrace{Df(\vec{c}(t_0))}_{\substack{1\times 3\\ \text{matrix}}}\cdot \underbrace{\vec{c}\,'(t_0)}_{\substack{3\times 1\\ \text{vector}}} \quad \text{at } t_0\in\mathbb{R}.$$

or as functions: $$\underbrace{\frac{d}{dt}(f\circ\vec{c})(t)}_{\substack{\text{expression in}\\ t}} = \underbrace{Df(\vec{c}(t))}_{\substack{1\times 3 \text{ matrix}\\ \text{of functions}\\ \text{in } t}}\cdot \underbrace{\vec{c}\,'(t)}_{\substack{3\times 1\\ \text{vector of}\\ \text{functions}\\ \text{in } t.}}$$

Here the right hand side looks like

$$Df(\vec{c}(t))\cdot\vec{c}\,'(t) = \left[\frac{\partial f}{\partial x}(\vec{x}),\ \frac{\partial f}{\partial y}(\vec{x}),\ \frac{\partial f}{\partial z}(\vec{x})\right]\Bigg|_{\vec{x}=\vec{c}(t)} \cdot \begin{bmatrix} x'(t)\\ y'(t)\\ z'(t)\end{bmatrix}$$

$$= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

And if the left hand side uses $h(t) = (f\circ\vec{c})(t)$, then

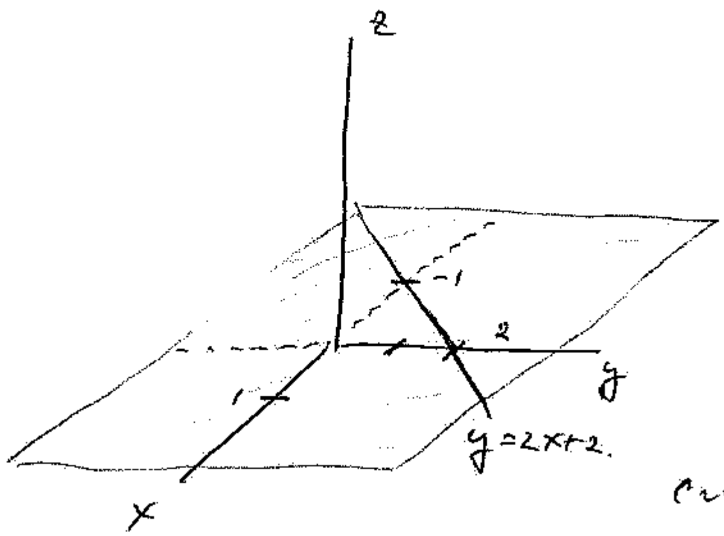
$$\frac{d}{dt}h(t) = \frac{dh}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

just like the Chain Rule (in Leibniz notation) in SVC.

ex. with functions

Let $f(x,y,z) = x^2+y^2+z^2$, where level sets of f are concentric spheres of increasing radius (as f increases) in $\mathbb{R}^3$.

Let $\vec{c}(t) = \begin{bmatrix} t \\ 2t+2 \\ 0 \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$. Here, $\vec{c}(t)$ is a parameterization of the line in the xy-plane in xyz-space with equation $y = 2x+2$:



Q: How is f changing along $\vec{c}$?

First, we construct $h(t) = (f \circ \vec{c})(t)$ and then take its derivative:

$$h(t) = (f \circ \vec{c})(t) = f(\vec{c}(t)) = (x^2+y^2+z^2)\Big|_{\vec{x}=\vec{c}(t)}$$

$$= (t)^2 + (2t+2)^2 + (0)^2$$

$$= t^2 + 4t^2 + 8t + 4 = 5t^2 + 8t + 4.$$

And so $h'(t) = \frac{d}{dt} h(t) = 10t + 8$.

Second, we recalculate $h'(t)$ via the chain Rule:

$$\frac{d}{dt}h(t) = \left.\frac{\partial f}{\partial x}\right|_{\vec{x}=\vec{c}(t)} \cdot \frac{dx}{dt} + \left.\frac{\partial f}{\partial y}\right|_{\vec{x}=\vec{c}(t)} \cdot \frac{dy}{dt} + \left.\frac{\partial f}{\partial z}\right|_{\vec{x}=\vec{c}(t)} \cdot \frac{dz}{dt}$$

$$= 2x\Big|_{\vec{x}=\vec{c}(t)} \frac{d}{dt}(t) + 2y\Big|_{\vec{x}=\vec{c}(t)} \cdot \frac{d}{dt}(2t+2) + 2z\Big|_{\vec{x}=\vec{c}(t)} \cdot \frac{d}{dt}(c)$$

$$= 2(t)(1) + 2(2t+2)(2) + 0$$

$$= 4t + 8t + 8 = 10t + 8.$$

Now look at the Mathematica Code to see when the dot on $\vec{c}(t)$, when moving, passes through decreasing level sets of f and when increasing. Match this with the sign of $h'(t) = 10t+8$.

<u>Another ex</u> Suppose $\vec{f}: \mathbb{R}^3 \to \mathbb{R}^2$, $\vec{g}: \mathbb{R}^4 \to \mathbb{R}^3$ are C^1 everywhere. Then $(\vec{f} \circ \vec{g}): \mathbb{R}^4 \to \mathbb{R}^2$ is differentiable. Can we calculate its derivative in terms of the derivatives of $\vec{f}$ and $\vec{g}$?

Sure. Create $\bar{h}(\vec{x}) = (\bar{f} \circ \bar{g})(\vec{x})$, so
$\bar{h}: \mathbb{R}^4 \to \mathbb{R}^2$, and $h(x,y,z,w) = f(g(x,y,z,w))$

Call the outputs of g by the variables r, s, t.

Then $g(x,y,z,w) = (r(x,y,z,w), s(x,y,z,w), t(x,y,z,w))$

These are the inputs to the outer function f, so that

$$h(x,y,z,w) = f\left(r(x,y,z,w), s(x,y,z,w), t(x,y,z,w)\right)$$

and
$$\underbrace{D\vec{h}(\vec{x})}_{\substack{2\times 4 \\ \text{matrix}}} = \underbrace{D\vec{f}(\vec{g}(\vec{x}))}_{\substack{2\times 3 \\ \text{matrix}}} \cdot \underbrace{D\bar{g}(\vec{x})}_{\substack{3\times 4 \\ \text{matrix}}}$$

$$\begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} & \frac{\partial r}{\partial w} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} & \frac{\partial s}{\partial z} & \frac{\partial s}{\partial w} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} & \frac{\partial t}{\partial w} \end{bmatrix}$$

By matrix multiplication, for example,
$$\frac{\partial h_1}{\partial y} = \frac{\partial f_1}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial f_1}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial y}$$

All other entries are similar.