Lecture 9: ~~[illegible]~~ V

Some data associated with derivatives of functions:

p. 111-112

(I) If a function $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at a pt $\vec{x}_0 \in U$, then the derivative of f at $\vec{x}_0$ is the $1 \times n$ row matrix

$$Df(\vec{x}_0) = \left[\frac{\partial f}{\partial x_1}(\vec{x}_0), \frac{\partial f}{\partial x_2}(\vec{x}_0), \cdots, \frac{\partial f}{\partial x_n}(\vec{x}_0)\right].$$

(There are reasons for this. For now, it makes vector computations easier. You'll see.

(II) In the special case where $n=2$, if f is differentiable at $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$, then the equation

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$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

is called the tangent plane of the graph(f) at the pt $(x_0, y_0, f(x_0, y_0)) \in \mathbb{R}^3$.

Notes: (a) There is a serious reason why this is true. You will know this once we understand better what is going on.

(b) ex. $f(x,y) = \cos(xy) + x^2 y$.

Find the plane tangent to graph(f) at $(x_0, y_0) = (1, \pi)$.

Strategy: Direct calculation using (II) above.

Solution: For
$$z = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](x-x_0),$$

we need

- $$f(x_0,y) = \cos(1(\pi)) + 1^2 \cdot \pi = \pi - 1$$

- $$\frac{\partial f}{\partial x}(x,y) = -y(\sin xy) + 2xy$$
$$\frac{\partial f}{\partial x}(1,\pi) = -\pi(\sin \pi) + 2\pi = 2\pi$$

- $$\frac{\partial f}{\partial y}(x,y) = -x(\sin xy) + x^2$$
$$\frac{\partial f}{\partial y}(1,\pi) = -\sin\pi + 1 = 1.$$

Solution (cont'd).

$$\text{So} \quad z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)$$

$$= \pi - 1 + [2\pi](x-1) + [1](y-\pi)$$

$$= \pi - 1 + 2\pi x - 2\pi + y - \pi$$

$$\boxed{z = -1 - 2\pi + 2\pi x + y}$$

and this is the equation of the plane tangent to $f(x,y)$ at $(x_0,y_0) = (1,\pi)$.

Or if one prefers,

$$\underbrace{2\pi}_{A=2\pi} x + \underbrace{y}_{B=1} \underbrace{- z}_{C=-1} \underbrace{-1-2\pi}_{D=-1-2\pi} = 0$$

for the equation of a plane in $\mathbb{R}^3$. ∎

III Note by I above, $Df(\vec{x}_0)$ is an $1 \times n$ matrix. We also define the $n \times 1$ matrix $(Df(\vec{x}_0))^T$ as the gradient of f at $\vec{x}_0$;

p. 112

$$\operatorname{grad} f = \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = (Df(\vec{x}_0))^T.$$

IV

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable.

Then $Df(\vec{x}_0)$ is an $m \times n$ matrix:

p. 111

$$Df(\vec{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where $f(\vec{x}_0) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$ and all partials are evaluated at $\vec{x}_0$.

ex. Find the matrix of partials for

(2.3.9(b)) $$f(x,y) = (xe^y + \cos y,\ x,\ x + e^y)$$

and evaluate at $(2,0)$.

Solution: $Df(\vec{x})$ is a 3×2 matrix since $f : \mathbb{R}^2 \to \mathbb{R}^3$. Here

$$Df(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} e^y & xe^y - \sin y \\ 1 & 0 \\ 1 & e^y \end{bmatrix}$$

Is f diff. everywhere? Yes by above Thm.

Solution (cont'd)

and at $(x_0, y_0) = (2, 0)$, we get

$$Df(2,0) = \begin{bmatrix} e^0 & 2e^0 - \sin 0 \\ 1 & 0 \\ 1 & e^0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(V) Just like Calculus 1:

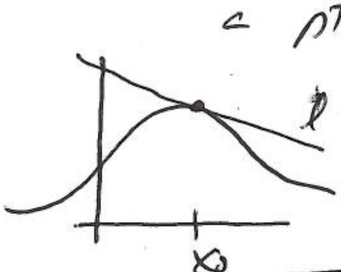
Thm 8 p.113.

Thm Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\bar{x}_0 \in U$. Then f is continuous @ $\bar{x}_0 \in U$.

(VI) One use of the derivative?

Recall, the tangent line to the graph ($f: \mathbb{R} \to \mathbb{R}$) at a pt

p.110

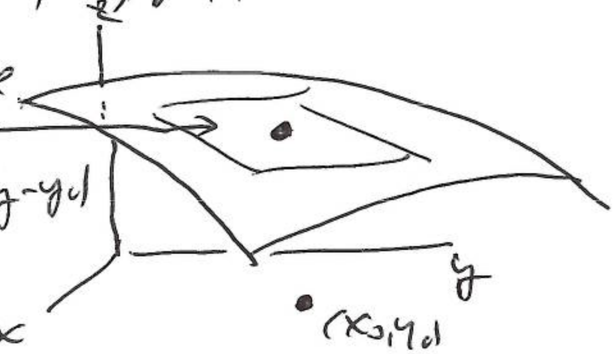


$l: y - y_0 = f'(x_0)(x - x_0)$, or

$y = y_0 + f'(x_0)(x - x_0)$

l is the best linear approx to graph(f) at x_0.

Is there a best linear approx to graph ($f: \mathbb{R}^2 \to \mathbb{R}$) at (x_0, y_0)? Yes, the tangent plane



$$f(x,y) - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

or

$$z = f(x_0, y_0) + Df(x_0, y_0)\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.$$

matrix mult

A special type of function

Given $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, choose $n = 1$ and let $m > 1$. This kind of function has a name:

Def A path (or curve) in $\mathbb{R}^n$ is a continuous function (we call a continuous function a map)

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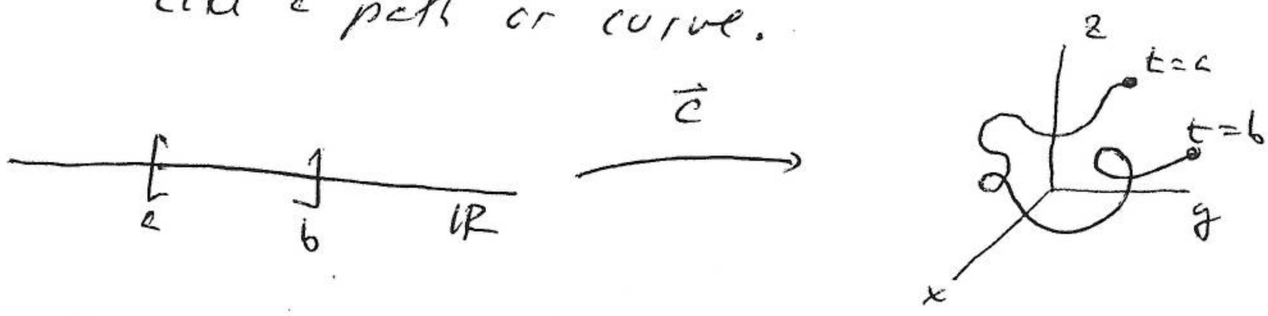
$$\vec{c} : [a, b] \to \mathbb{R}^n, \quad \vec{c}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Notes ① It is called a path or a curve because its range (or image) inside $\mathbb{R}^n$ looks like a path or curve.



② It is a parameterized curve in $\mathbb{R}^n$ because the input variable plays the role of a parameter (coordinate t directly on curve).

Notes cont'd ③ The domain of $\vec{c}$ may also be open interval (a,b), or all of $\mathbb{R}$.

ex. From Exercise 1.2.26, the line given by

$$x = -1+t,\ y = -2+t,\ z = -1+t$$

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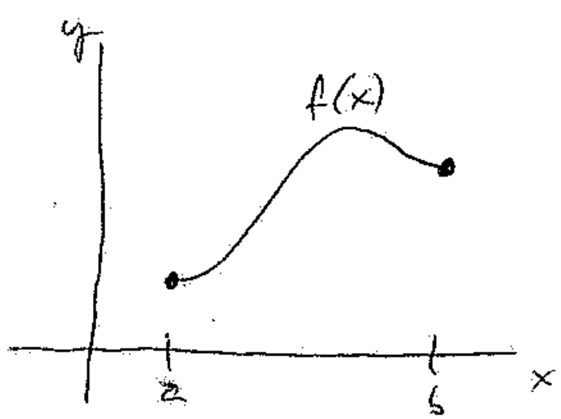
is actually a path in $\mathbb{R}^3$, where

$$\vec{c} : \mathbb{R} \to \mathbb{R}^3,\quad \vec{c}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -1+t \\ -2+t \\ -1+t \end{bmatrix}$$

④ One quick way to generate curves in $\mathbb{R}^2$:

ex. Take any function $f : [a,b] \to \mathbb{R}$ from SVC. Rewrite it as

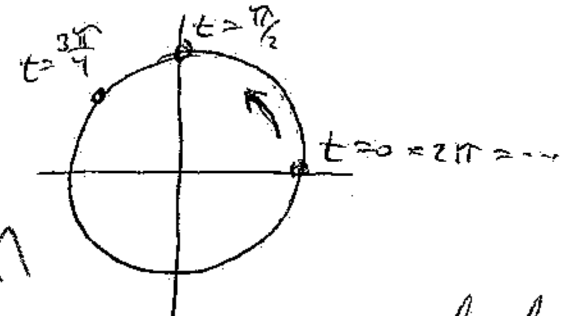
$$\vec{c} : [a,b] \to \mathbb{R}^2,\quad \vec{c}(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix}$$



Then its graph looks like a curve in $\mathbb{R}^2$ parameterized by x. (the horizontal coordinate).

(5) All curves above looked to be 1-1, where 1-1 means that if $\vec{c}(t_1)=\vec{c}(t_2)$ then $t_1=t_2$. In other words, the curve in $\mathbb{R}^n$ never crosses itself or retraces its steps. But many curves actually do cross itself and/or retrace its steps!

ex. Let $\vec{c}:\mathbb{R}\to\mathbb{R}^2$, $\vec{c}(t)=(\cos t, \sin t)=\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}$.



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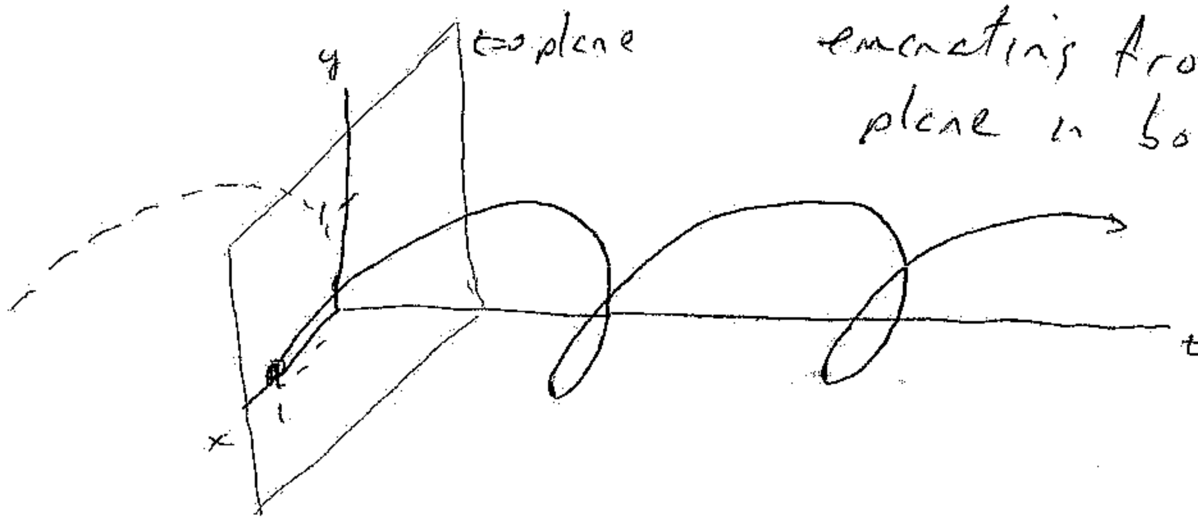
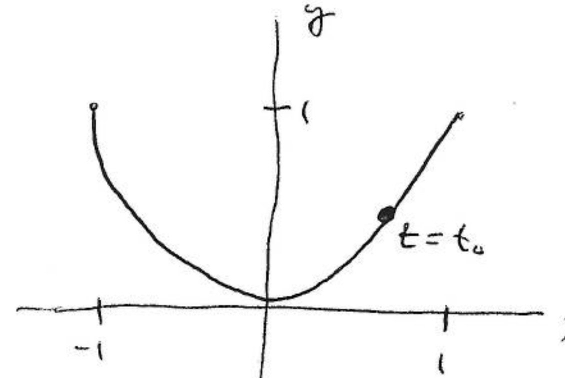
This path infinitely winds $\mathbb{R}$ around the unit circle. Note that the actual graph of $\vec{c}$ "lives" in $\mathbb{R}^3$, seen as the txy-space. It looks like a spiral emanating from the t=0 plane in both directions



Notes (5) cont'd

Here the graph is nice but the parameterized curve $\vec{c} \subset \mathbb{R}^2$ is easier to "see":

ex. Let $\vec{c}(t) = \begin{bmatrix} \sin t \\ \sin^2 t \end{bmatrix}$.

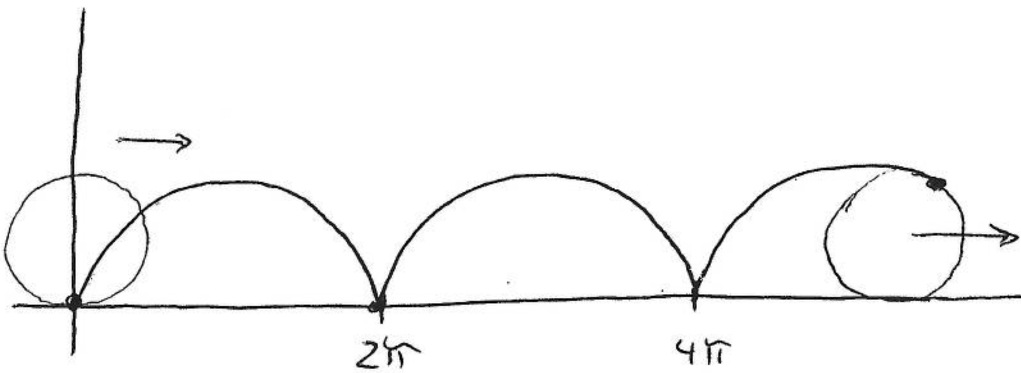


p. 118 Then $\vec{c}(t)$ traces out this parabolic shape over and over again, backwards and forwards. What happens when the "bead" traveling along the wire at time t approaches $(-1,1)$ or $(1,1)$?

ex. A strange 1-1 curve is the cycloid, $\vec{c}: \mathbb{R} \to \mathbb{R}^2$, $\vec{c}(t) = (t - \sin t, 1 - \cos t)$. It

p. 119 traces out the curve given by a point on the edge of a wheel as the wheel rolls along the x-axis:

XIV



Q: What happens at $t = 2\pi, 4\pi, \dots, 2n\pi, \dots$

⑥ If $\vec{c}:[a,b] \to \mathbb{R}^n$ is differentiable, then

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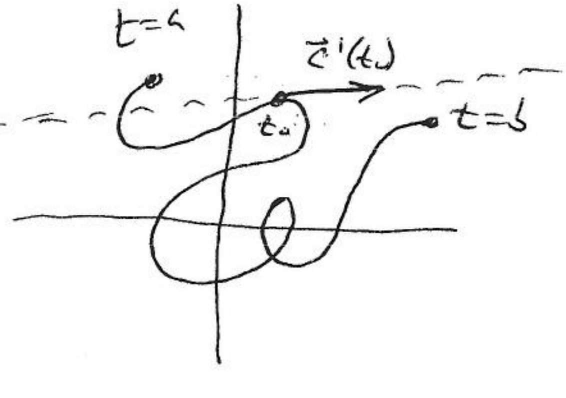
$$D\vec{c}(t) = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$ is again a n-vector

At $t=t_0$, $D\vec{c}(t_0) = \vec{c}\,'(t_0)$ is the vector in $\mathbb{R}^n$ based at $\vec{c}(t_0)$. It is tangent to $\vec{c}(t)$ at $t=t_0$, and when $\vec{c}\,'(t_0) \neq 0$, define

p.122.

$$l(t) = \vec{c}(t_0) + \vec{c}\,'(t_0)(t-t_0)$$

as the line tangent to $\vec{c}(t)$ at $t=t_0$.



Here $\vec{c}\,'(t_0)$ is called the velocity of the curve at $t=t_0$ (for obvious reasons).

Final question: Are the curves given by $\vec{c}=\begin{bmatrix}\sin t\\ \sin^2 t\end{bmatrix}$, and the cycloid differentiable curves? Important function.

Yes, since the two vectors $\vec{c}=\begin{bmatrix}\sin t\\ \sin^2 t\end{bmatrix}$ and $\vec{c}(t)=\begin{bmatrix}t-\sin t\\ 1-\cos t\end{bmatrix}$ are differentiable everywhere.

Their images in $\mathbb{R}^2$ are not, though.

This is not contradictory since in the parameterization given by the functions,

$$\vec{c}\,'(t)=\begin{bmatrix}\cos t\\ 2\sin t\cos t\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \text{ at } t=\tfrac{\pi}{2},\tfrac{3\pi}{2},\ldots$$

(when $\vec{c}(\tfrac{\pi}{2})=\begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{c}(\tfrac{3\pi}{2})=\begin{bmatrix}-1\\1\end{bmatrix}$.

and $\vec{c}\,'(t)=\begin{bmatrix}1-\cos t\\ \sin t\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ at $t=0,2\pi,4\pi$ ...

If a bead on a wire stops momentarily in its path, it can change direction differentiably, even if its path has a corner there.