Notes ① If a limit exists, it is unique!

② All of the techniques and properties of limits hold from 1-dim calculus. See text.

③ Many techniques in actual calculation use properties of limits to evaluate.

Others involve reducing to 1-dim calculus.

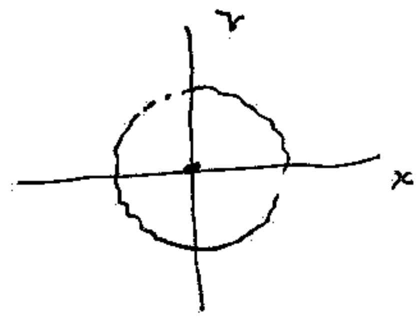
ⓐ Functions you know have limits still do. (polynomials, trig fncs, ...)

ex. $\lim_{(x,y)\to(x_0,y_0)} \cos(xy)$

$\lim_{(x,y)\to(x_0,y_0)} x^2+y^2$

ⓑ Approaching $\vec{x}_0$ from particular directions makes for a 1-d limit.

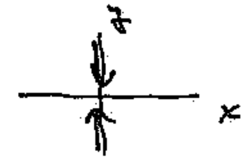
ex. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = ?$



Here domain is 2-d

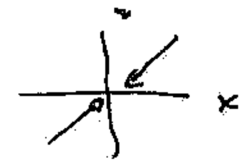
To test for a limit, come in from different directions.

- Along the line $x = 0$?



Here $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0y}{0^2+y^2} = 0$

- Along the line $y = x$?



$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Since $\frac{1}{2} \neq 0$, this limit does not exist.

(c) Conversion to polar

With $x = r\cos\theta$, $y = r\sin\theta$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{r^2}$$

$$= \lim_{r\to 0} \cos\theta\sin\theta$$

And since for different approaches to 0 along different lines $\theta = \text{constant}$, we would get different answers, limit does not exist.

Continuity is similar to that of Calc I:

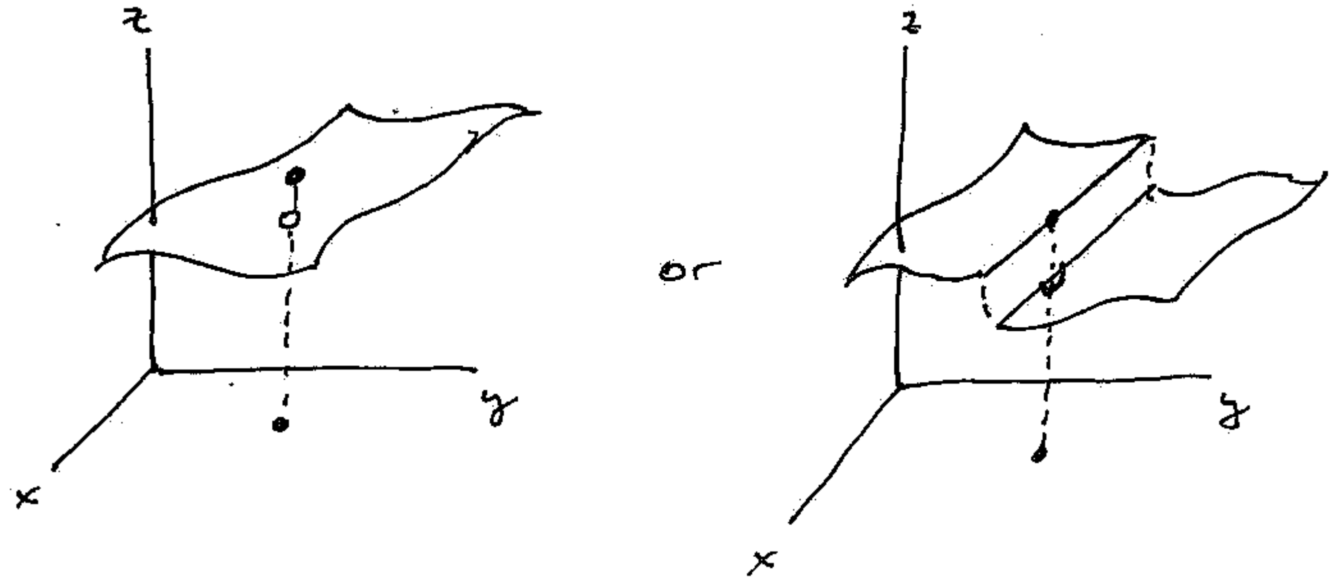
$f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\vec{x}_0 \in U$ if

(I) $\lim_{\vec{x}\to\vec{x}_0} f(x)$ exists, and

(II) $f(\vec{x}_0)$ is defined, and

(III) $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$.

Visually continuity fails in the obvious way:



Notes ① All limit laws are similar (in text) to 1-d.

② All properties of continuous fncs are similar also.

③ Plus: $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, where $f(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$ is continuous iff each $f_i : A \subset \mathbb{R}^n \to \mathbb{R}$ is cont.

④ Even in multiple variables, polynomials, exponentials and logarithms, rational fncs, trig fncs, etc are all continuous on their domain.

ex. $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (\cos(yz), ye^x)$ is continuous everywhere. (why?)