Lecture 6: Section 2.2.

Q: So what is the <u>limit</u> of a function at a pt?

Intuition

A: In general, it is a statement (property) of a fnc?

A function f has a limit L at a pt a

if ~~[illegible]~~ at all pts in the domain of f

"near" a, have function values "near" L.

- "near" in the sense of "as close as one can get".
- A limit conveys information about a fnc <u>at a pt</u> about how f behaves near the pt.

Let ~~[illegible]~~ $f: X \subset \mathbb{R} \to \mathbb{R}$

Precise

A: f has a limit L at a pt $x = c$ in the domain of f if for every $\varepsilon > 0$, there is a $\delta > 0$ so that whenever $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

If L exists, we write $\lim\limits_{x \to c} f(x) = L$.

We talked in the last lecture about neighborhoods of points, and small neighborhoods like the set

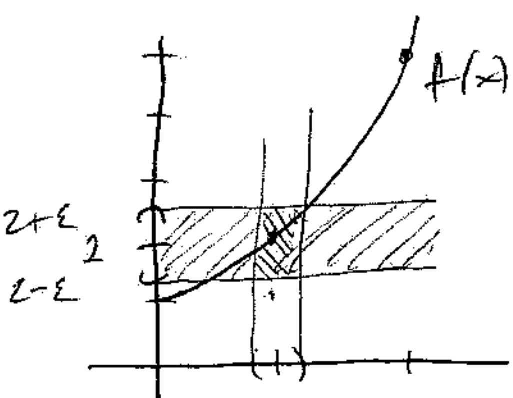
$$B_\delta(a) = \{ x \in \mathbb{R} \mid |x-a|<\delta \}.$$

Here, in the above definition

$0<|x-a|<\delta$ is the condition $x \in B_\delta(a)$ but $x \neq a$.

and $|f(x)-L|<\varepsilon$, is the condition $f(x) \in B_\varepsilon(L)$.

ex. From Calc I, Let $f(x) = x^2+1$, on $[0,2]$.



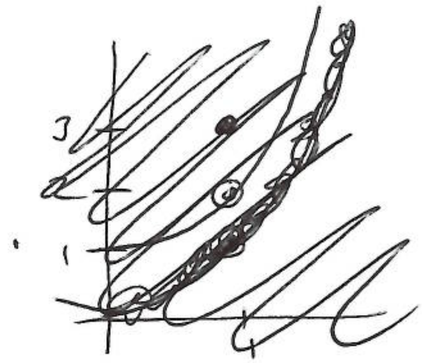
Here we know $\lim_{x\to 1} f(x) = 2$.

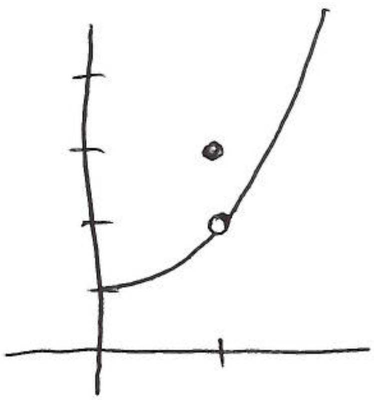
Why? Because given any small number $\varepsilon>0$, I can form a small ε-neighborhood $B_\varepsilon(2)$ around 2 which looks like the horizontal band at left. If given this band, I can find a $\delta>0$ where every pt $x \in B_\delta(1)$ has image $f(x) \in B_\varepsilon(2) = (2-\varepsilon, 2+\varepsilon)$.

And if this works for <u>any</u> $\varepsilon>0$, then $\lim_{x\to 1} f(x) = 2$.

Some Notes ① It doesn't matter what $f(a)$ is or even if it is defined for the limit to exist. That is why the condition includes the $0<|x-a|<\delta$ part.

ex. Let $g(x)=\begin{cases} x^2+1 & x\neq 1 \\ 3 & x=1 \end{cases}$





Here $\lim\limits_{x\to 1} g(x)=2$

even if $g(1)\neq 2$. Remember?

② How can a limit fail to exist?

ⓐ if a function has a jump discontinuity is one way.

ex. Let $h(x)=\begin{cases} -1 & x<0 \\ 1 & x\geq 0 \end{cases}$

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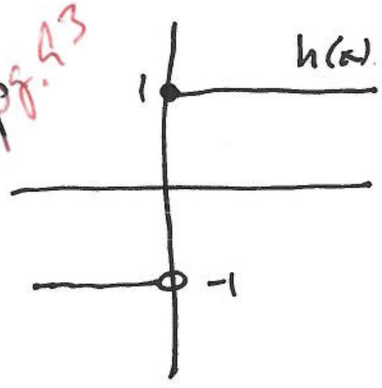

Here, $h(0)=1$, but $\lim\limits_{x\to 0} h(x)$ does not exist since for any $\varepsilon>0$ and less than 2, there will always be pts x in any δ-ball around $x=0$ whose function values are not in $(2-\varepsilon, 2+\varepsilon)$.

For functions $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$, the concept is the same, although the open sets, the neighborhoods, are bigger dimensional sets.

<u>Def.</u> f has a limit $\vec{L} \in \mathbb{R}^m$ at a point $\vec{x}_0 \in \mathbb{R}^n$ written $\lim_{\vec{x} \to \vec{x}_0} \vec{f(\vec{x})} = \vec{L}$ if given any $\varepsilon > 0$, there is a $\delta > 0$ where if $\underbrace{0 < \| \vec{x} - \vec{x}_0 \| < \delta}_{\vec{x} \in B_\delta(\vec{x}_0)}$, then $\underbrace{\| \vec{f}(\vec{x}) - \vec{L} \| < \varepsilon}_{\vec{f(\vec{x})} \in B_\varepsilon(\vec{L})}$.

but $\vec{x} \neq \vec{x}_0$.

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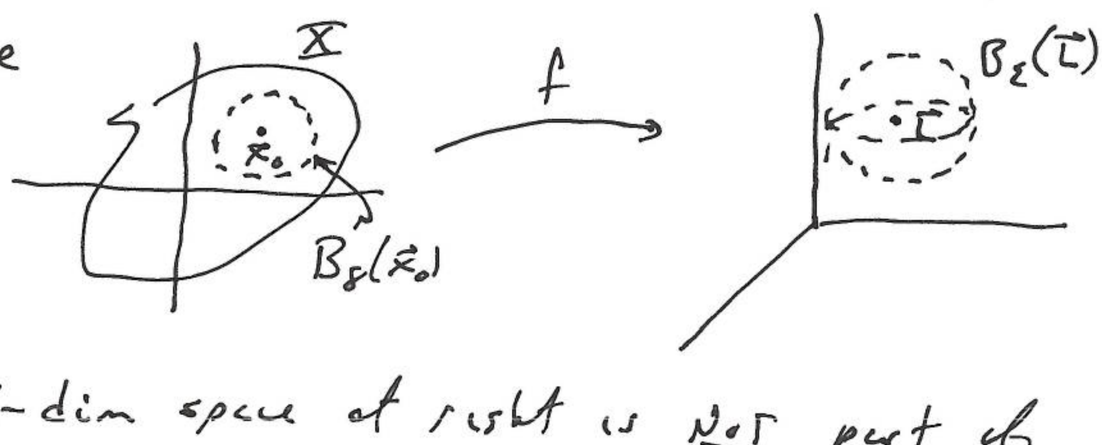
ex. if $f: X \subset \mathbb{R}^2 \to \mathbb{R}^3$, then the situation looks like



Note: The 3-dim space at right is <u>NOT</u> part of the graph(f), since that would live in $\mathbb{R}^5$. It is just the function values (and not the inputs also).

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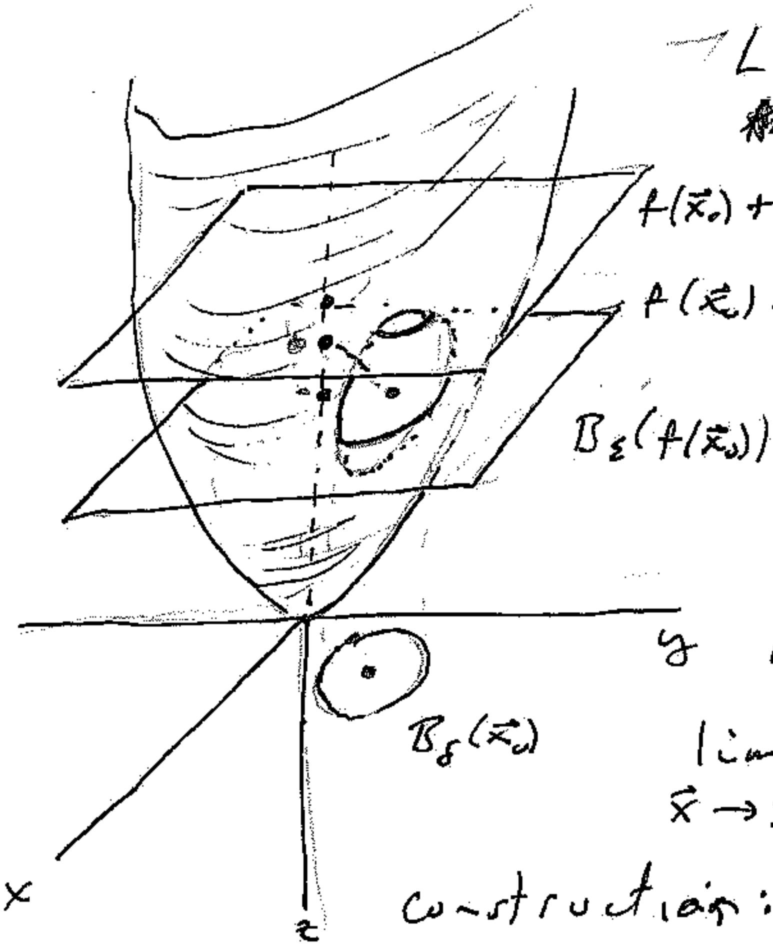
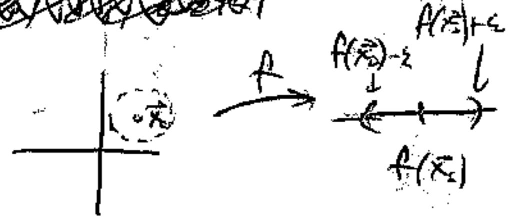
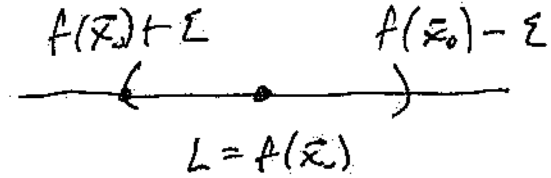




Here the graph of something like $z=x^2+y^2$ is shown. We check that $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$ here by construction: First, choose an $\varepsilon>0$ and form an $B_\varepsilon(f(\vec{x}_0))$. This is an interval in the output z-variable



Next, we try to find a neighborhood $B_\delta(\vec{x}_0)$ about $\vec{x}_0$ in xy-plane all of whose values are in $B_\varepsilon(f(\vec{x}_0))$. For all pts $\vec{x}\in B_\delta(\vec{x}_0)$, to be in $B_\varepsilon(f(\vec{x}_0))$, the graph of pts in $B_\delta(\vec{x}_0)$ would all have to be inside the 2 planes at $z=f(\vec{x}_0)+\varepsilon$ and $z=f(\vec{x}_0)-\varepsilon$.

This choice of δ fails above. But a smaller one would work.

Notes ① If a limit exists, it is unique!

② All of the techniques and properties of limits hold from 1-dim calculus. See text.

③ Many techniques in actual calculation use properties of limits to evaluate. Others involve reducing to 1-dim calculus.

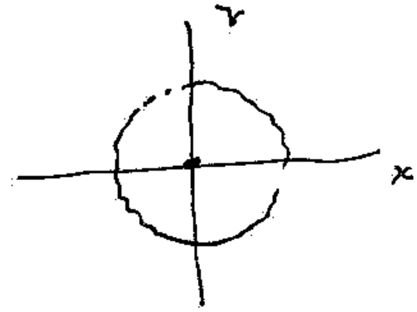
ⓐ Functions you know have limits still do. (polynomials, trig fncs, ...)

ex. $\lim_{(x,y)\to(x_0,y_0)} \cos(xy)$

$\lim_{(x,y)\to(x_0,y_0)} x^2+y^2$

ⓑ Approaching $\vec{x}_0$ from particular directions makes for a 1-d limit.

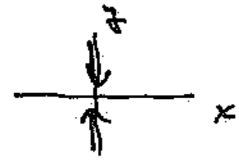
ex. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = ?$



Here domain is 2-d

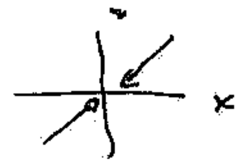
To test for a limit, come in from different directions.

- Along the line $x = 0$?



Here $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \frac{0y}{0^2+y^2} = 0$

- Along the line $y = x$?



$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Since $\frac{1}{2} \neq 0$, this limit does not exist.

(c) Conversion to polar

With $x = r\cos\theta$, $y = r\sin\theta$.
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{r^2}$$
$$= \lim_{r\to 0} \cos\theta \sin\theta$$

And since for different approaches to 0 along different lines $\theta =$ constant, we would get different answers, limit does not exist.

Continuity is similar to that of Calc I:

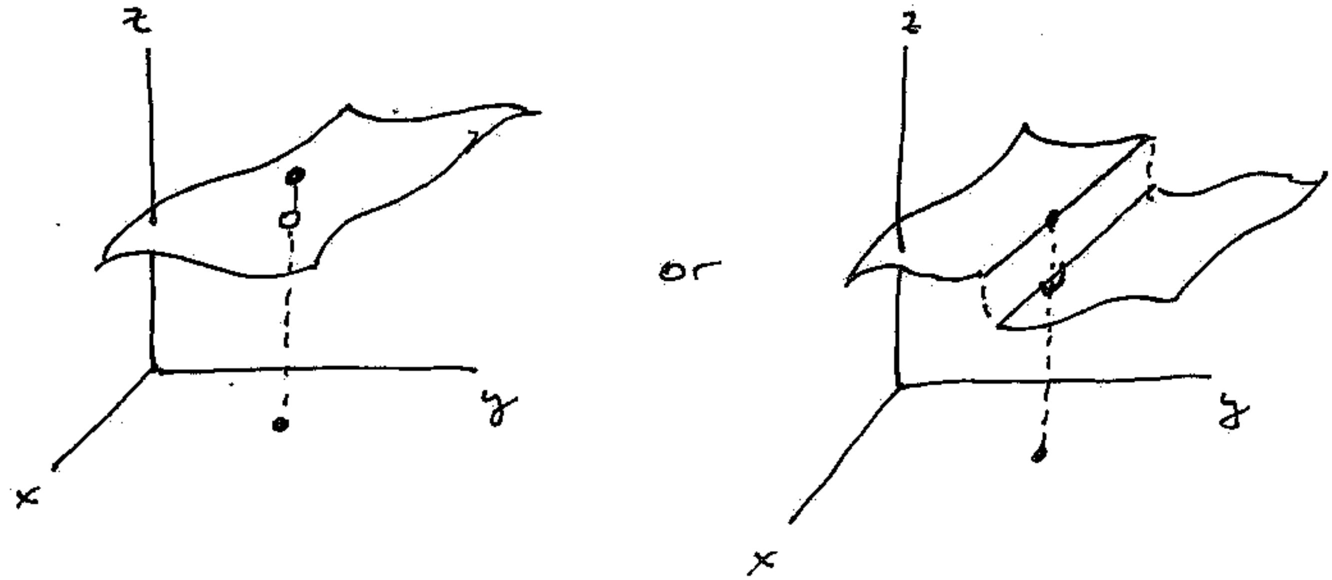
~~Def~~ $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\vec{x}_0 \in U$ if

(I) $\lim\limits_{\vec{x}\to\vec{x}_0} f(x)$ exists, and

(II) $f(\vec{x}_0)$ is defined, and

(III) $\lim\limits_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$.

Visually continuity fails in the obvious way:



Notes ① All limit laws are similar (in text) to 1-d.

② All properties of continuous fncs are similar also.

③ Plus: $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, where $f(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$ is continuous iff each $f_i : A \subset \mathbb{R}^n \to \mathbb{R}$ is cont.

④ Even in multiple variables, polynomials, exponentials and logarithms, rational fncs, trig fncs, etc are all continuous on their domain.

ex. $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x,y,z) = (\cos(yz), ye^x)$ is continuous everywhere. (why?)