Lecture 5: ~~[illegible]~~

Section 2.2.

<u>Def</u> An open disk (or open ball, the more common term) of radius $r>0$, in $\mathbb{R}^n$, centered at $\vec{x}_0$, is defined as

p. 88

$$B_r(\vec{x}_0) = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{x}_0\| < r \}$$

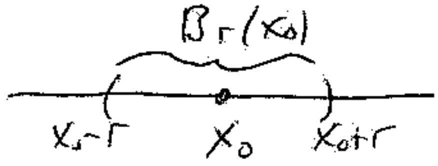
<u>Notes</u> (1) Most books use B for ball in this notation. This book uses D for disk. In either case, B for ball is used in all dimensions:

- a 2-dim ball is a disk,
- a 1-dim ball is an interval
- an n-dim ball is the n-dim analog of the standard 3-dim one.

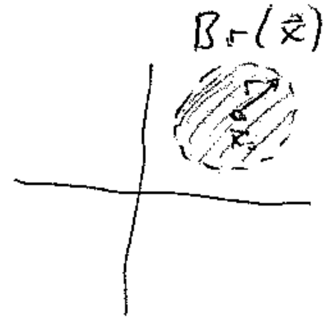
(In math, we use volume for size in $\mathbb{R}^3$. We also use volume for size in $\mathbb{R}^n$, $n>3$, and for $\mathbb{R}^2$ (area), and $\mathbb{R}$ (length). We use Ball in the same way).

cont'd.

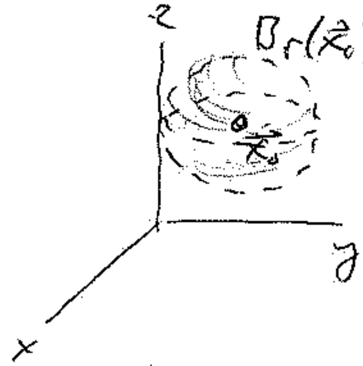
Notes (2) Visually



1-dim case



2-dim case



3-dim case

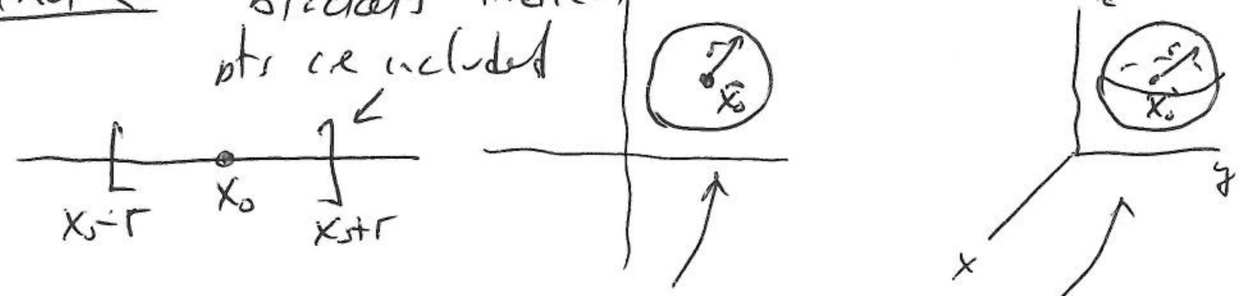
The dotted line is the 2-dim and higher dim way of denoting that the "edges" are <u>not</u> part of the set (notice the strictly inequality sign in the definition)

(3) The closed ball $\overline{B_r}(\vec{x}_0)$ is just the open ball $B_r(\vec{x}_0)$ along with its "edges":

$$\overline{B_r}(\vec{x}_0) = \{ \vec{x} \in \mathbb{R}^n \mid \| \vec{x} - \vec{x}_0 \| \leq r \}$$

Here, the "$\leq$" means include also the points exactly equal to an r-distance away from $\vec{x}_0$: the circle of radius r, centered at $\vec{x}_0$ in $\mathbb{R}^2$. The sphere of radius r in $\mathbb{R}^3$, the endpts of the interval in $\mathbb{R}$, etc.

Denoted — brackets indicate pts are included



Solid lines indicate the edges are included

Note: Sometimes, when not confusing, we will draw balls with solid lines even when open. It is easier.

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④ Pointwise, we would write $B_r((x_1, \dots, x_n))$.

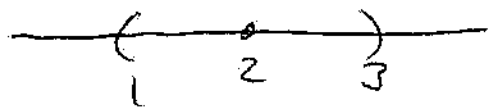
Def An arbitrary set $U \subset \mathbb{R}^n$ is called *open* if for every point $\vec{x} \in U$, there is some $\varepsilon > 0$, where $B_\varepsilon(\vec{x}_0)$ lies entirely inside U, so $B_\varepsilon(\vec{x}) \subset U$.

$U \subset \mathbb{R}^2$

p. 88

It doesn't matter how small ε is, as long as it can be positive for every pt $\vec{x} \in U$.

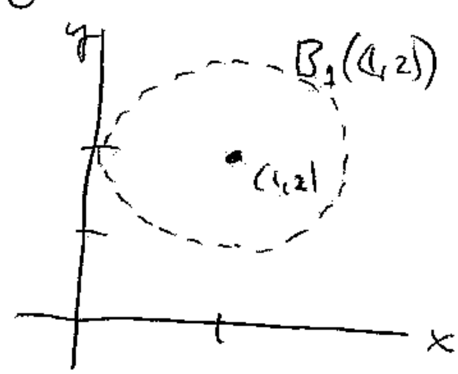
ex 1. The open ball $B_1(2) \subset \mathbb{R}$ is an open set, since for any pt $x \in B_1(2)$, the distance between x and the closest edge is positive.



~~Also~~ Call this distance r. Then the ball of ~~size~~ radius $\frac{r}{2}$ lies entirely inside $B_1(2)$.

Draw this!

ex 2: $B_1((1,2))$ is the interior of the ~~dashed~~ circle of radius 1 centered at $(1,2)$ in $\mathbb{R}^2$. The radius 1-circle centered at $(1,2)$ is NOT in $B_1((1,2))$.



Q: What is $B_1((1,2)) \cap \{y\text{-axis}\}$? A: It is ϕ.

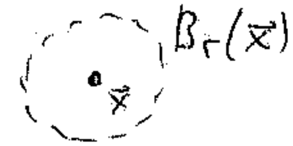
Q2: What is the distance between $B_1((1,2))$ and the y-axis?

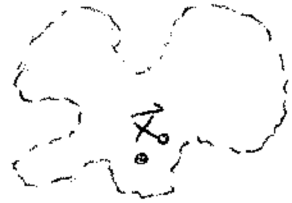
Recall the distance between 2 sets is defined as the minimum distance between any 2 pts, 1 in each set (like billiard balls the distance is 0 if they are touching).

A2: It is 0! They are not touching, but there is no distance between them.

~~Def: A pt $\vec{x} \in \mathbb{R}^n$ is called a boundary pt of a set $U \subset \mathbb{R}^n$ if~~

Note: Any open set in $\mathbb{R}^n$ containing $\vec{x} \in \mathbb{R}^n$ is called a neighborhood of $\vec{x} \in \mathbb{R}^n$.

ex. 

or 

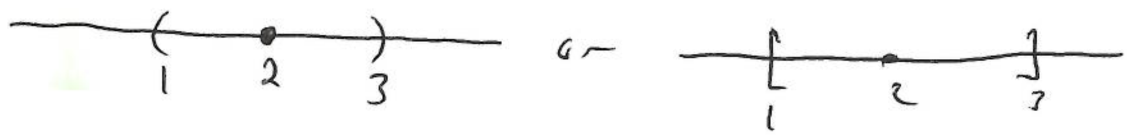
are neighborhoods of $\vec{x} \in \mathbb{R}^2$

Open balls are examples of neighborhoods of points but any open set is a neighborhood of its points.

<u>Def</u> A pt. $\vec{x} \in \mathbb{R}^n$ is called a <u>boundary</u> pt of a set $U \subset \mathbb{R}^n$ if every neighborhood of $\vec{x} \in \mathbb{R}^n$ contains at least 1 pt in U and one pt outside of U.

pg 90

ex. The endpts of an interval, whether in the interval or not, are boundary pts.



Here, $x=1$ and $x=3$ are the boundary pts of each of these. why?

ex. The circle of radius 1 centered at $(1,2)$ is the set of boundary pts (or the boundary) of $B_1((1,2)) \subset \mathbb{R}^2$.