Lecture 3: [redacted]

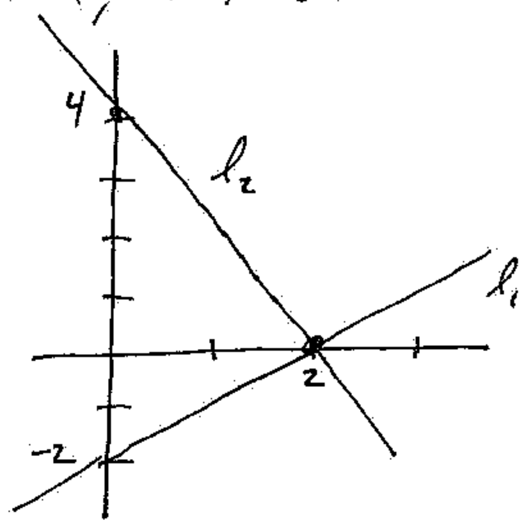
So how does one describe a line in $\mathbb{R}^3$?

An easier question: what is the equation for a pt. in $\mathbb{R}$? A: $x = a$, some constant.

What is the equation for a pt in $\mathbb{R}^2$?

A: There isn't one.

But any 2 lines, not parallel, intersect in 1 point. Hence we can use the system of 2 line equations to denote a point:



l_1: $x - y + 2 = 0$

l_2: $2x + y - 4 = 0$

This system of 2 equations in 2 unknowns defines the pt $(2,0) \in \mathbb{R}^2$.

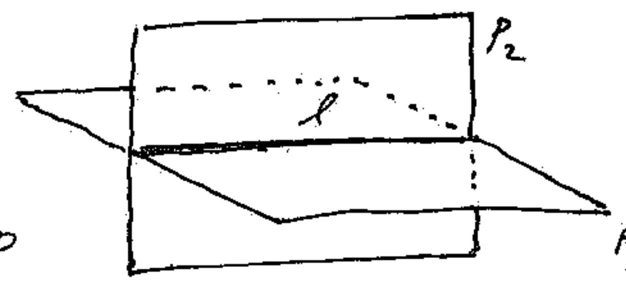
It is the set of all solutions to both equations together.

So again, what is the equation of a line in $\mathbb{R}^3$?

A: There isn't one.

Here are 2 ways to describe a line in $\mathbb{R}^3$, though:

(I) As the intersection of 2 planes



The line ℓ is the set of all solutions $(x, y, z) \in \mathbb{R}^3$ ~~that~~ to both planes equations:

$$P_1: \quad A_1 x + B_1 y + C_1 z + D_1 = 0$$
$$P_2: \quad A_2 x + B_2 y + C_2 z + D_2 = 0$$

Note: This is a system of 2 equations in 3 unknowns, with a "line" of solutions.

(II) Parametrically, with a coordinate defined directly on the line.

ex. Problem 1.2.26 uses the line

$$x = -1 + t, \quad y = -2 + t, \quad z = -1 + t$$

Note: Here this is actually a system of 3 equations in 4 unknowns. Do you see a pattern here?

So how does one go from a line defined by a system of 2 planar equations to the parameterized line?

Easy! The same techniques you used to "calculate" the pt. which is the intersection of 2 lines, you can use to find the parameterized line from the system of 2 planes.

Namely, in any system of equations, any linear combination of 2 equations is another equation that has the <u>same</u> solution set.

Chosen carefully, any new equations you create can be simpler (with less variables) than the originals.

The Mathematica page IntersectionOf2Planes.nb gives an example.

Here is another:

ex. Parameterize the line given by

$$4x - y + z = 4$$
$$2x + y - 3z = 6 \quad .$$

Strategy: Use equation arithmetic to ~~write~~ find new equations which all write some of the variables in terms of only 1. Use that variable as the parameter.

Solution: Call $4x-y+z=4$ $\boxed{Eq1}$ and the other $\boxed{Eq2}$.

Step 1: Add $\boxed{Eq1}$ to $\boxed{Eq2}$ to get

$$6x - 2z = 10, \text{ or } z = -5 + 3x$$

Step 2: Add $3\boxed{Eq1}$ to $\boxed{Eq2}$ to get $y = 7x - 9$.

Step 3: Letting $x = t$, we then get

$x = t$, $y = 7t - 9$, $z = -5 + 3t$ as our parameterization

Notes ⓐ. 2 equations in 2 unknowns can determine a pt in $\mathbb{R}^2$.

ⓑ 1 equation in 2 unknowns determines a line in $\mathbb{R}^2$

ⓒ 2 equations in 3 variables and 3 eqns in 4 variables can determine a line in $\mathbb{R}^3$.

Q: 1 equation in 2 variables determines what kind of space in $\mathbb{R}^3$? Careful: it is the number of eqns in the size of the space.

ⓓ Q: 3 equations in 3 variables? If none of the planes are parallel or trivial, this determines a pt in $\mathbb{R}^3$.

ex. The xy-plane, and the xz-plane, and the yz plane all intersect where?

Section 1.4 will be covered in the context of other parts of this course. We will come back to this later.

Section 1.5 deals with how things are defined and live in $\mathbb{R}^n$, $n \geq 1$.

It is here that:

(A) The dot product is defined in $\mathbb{R}^n$.

(B) The transpose of a matrix is introduced (though it is not named in the section, but only in an exercise in the back).

And it is here that the book does something silly:

On page 64, the book writes:

"Let $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Consider the $n \times 1$ ~~matrix~~ column matrix associated with $\vec{x}$, which we will denote $\vec{x}^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$."

Later they admit this is confusing:

"Thus, although this may cause some confusion, we will write $\vec{x} = (x_1, \cdots, x_n)$ and $\vec{y} = (y_1, \cdots, y_n)$ and when we need them for matrix calcs, we will write $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ without the T."

We will not do this! Instead, we will do:

1. Always consider $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ as a <u>point</u> in $\mathbb{R}^n$.
2. Always view $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ as a column vector
3. Always denote $\vec{x}^T = [x_1 \cdots x_n]$ as a row vector.

<u>Note</u> • There is a natural identification of pts $(x_1, \cdots, x_n) \in \mathbb{R}^n$ with vectors $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ since any pt can serve as the head of a vector with the same data and tail at the origin.

Note: • Thus it is not confusing to write:

$$\vec{x}\cdot\vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [x_1 \cdots x_n]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{x}^T\vec{y} = \sum_{i=1}^{n} x_i y_i.$$

And even if we cannot "see" vectors in $\mathbb{R}^n$, $n>3$, we can still see the angle between them in $\mathbb{R}^n$.

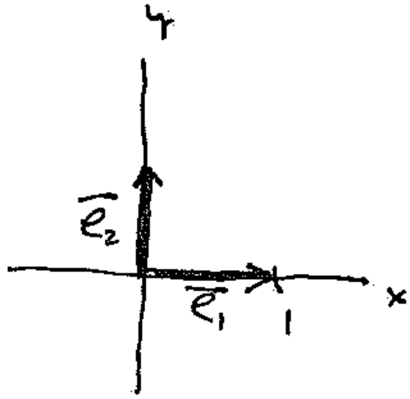
© We learn about the structure of vectors

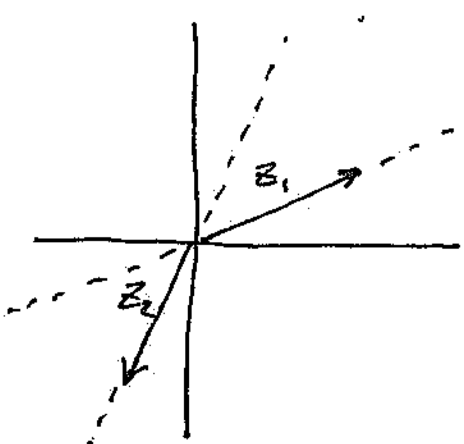
In $\mathbb{R}^2$, any vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ based at $(0,0) \in \mathbb{R}^2$ can be written as the sum

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\vec{\imath} + y\vec{\jmath}$$

Here, $\vec{\imath}, \vec{\jmath}$ are called basis vectors and <u>any</u> vector $\vec{v} \in \mathbb{R}^2$ can be written as a <u>linear combination</u> of these.

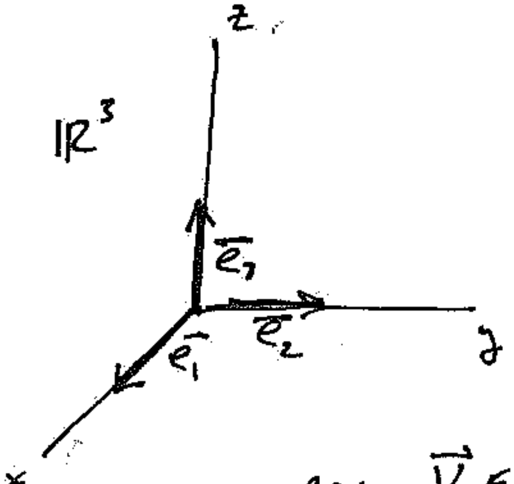
Notes
- We also call $\vec{\imath}, \vec{\jmath}$ by the name $\vec{e}_1, \vec{e}_2$.
- These 2 basis vectors are the "rulers" measuring unit length along the 2 axes. You will learn in linear algebra that
  - i) They are called standard basis vectors, and
  - ii) There is absolutely nothing special about them (apart from the accepted convention).





At left, the 2 dotted lines are perfectly acceptable as "axis" lines, with unit lengths given by basis vectors $\vec{z}_1, \vec{z}_2$. It simply makes for a different grid system on $\mathbb{R}^2$.

In $\mathbb{R}^n$, we play the same game:



$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \vec{e_n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

is a complete set of basis vectors in $\mathbb{R}^n$, so that for any $\vec{V} \in \mathbb{R}^n$,

$$\vec{V} = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = x_1 \vec{e_1} + x_2 \vec{e_2} + \cdots + x_n \vec{e_n}$$

And since for any i, and $j \neq i$, we get $\vec{e_i} \cdot \vec{e_j} = 0$. These basis vectors are all orthogonal to each other.

(D) We introduce linear systems

$$A \vec{x} = \vec{y}$$

where $A_{m \times n}$ is a m-row by n-column matrix.

This equation only makes sense if $\vec{x} \in \mathbb{R}^n$, and $\vec{y} \in \mathbb{R}^m$, since then

$$A_{m \times n} \vec{x}_{n \times 1} = \vec{y}_{m \times 1}$$

Let $f: \mathbb{R}^{\square} \longrightarrow \mathbb{R}^{\square}$ be a function from some Euclidean space to another, given by $f(\vec{x}) = A\vec{x}$ and $A_{m\times n}$ is a matrix.

Q: What numbers go in the 2 boxes? That is, how big is the domain space and how big is the range space?

A: Since $f(\vec{x}) = A\vec{x} = A_{m\times n}\vec{x}$, $\vec{x}$ must be $n\times 1$ as a vector. The result then is

$$\vec{y} = f(\vec{x}) \text{ where } y \text{ is } m\times 1.$$

Hence $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $f(\vec{x}) = A\vec{x} = \vec{y}$.

Note: When $m > 1$, the output $\vec{y} = f(\vec{x})$ is a vector. Hence we should write $\overrightarrow{f(\vec{x})}$, or

$$\vec{f}: \mathbb{R}^n \to \mathbb{R}^m, \quad \overrightarrow{f(\vec{x})} = A\vec{x}.$$

Sometimes we may leave this off to avoid too much notation.

Def Any function is called linear, if

$$f(a\vec{x}+b\vec{y}) = a f(\vec{x}) + b f(\vec{y}).$$

Functions defined by matrices are always linear.

ex. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not linear since

$$f(2+4) = f(6) = 36$$

$$\neq f(2) + f(4) = 4 + 16 = 20$$

ex. $g: \mathbb{R} \to \mathbb{R}$ $g(x) = 3x$ is linear since

$$g(3x_1 - 4x_2) = 3(3x_1 - 4x_2) = 9x_1 - 12x_2$$

$$\parallel$$

and $$3g(x_1) - 4g(x_2) = 3(3x_1) - 4(3x_2) = 9x_1 - 4x_2.$$

ex. $h: \mathbb{R} \to \mathbb{R}$ $h(x) = 3x + 5$ is NOT linear.

Why not? Because

$$h(3x_1 - 4x_2) = 3(3x_1 - 4x_2) + 5 = 9x_1 - 12x_2 + 5$$

but $$3h(x_1) - 4h(x_2) = 3(3x_1+5) - 4(3x_2+5)$$

$$= 9x_1 + 15 - 12x_2 - 20$$

$$= 9x_1 - 12x_2 - 5$$ "See?"