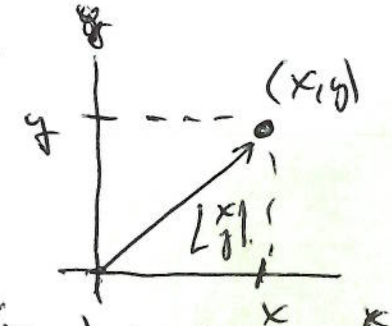
Lecture 2.

From SVC you understand that the plane

$$\mathbb{R}^2 = \{ (x,y) \mid x \in \mathbb{R},\ y \in \mathbb{R} \} = \mathbb{R} \times \mathbb{R}.$$

the set of all 2-tuples of real numbers, can also be represented as the set of all 2-vectors

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x \in \mathbb{R},\ y \in \mathbb{R} \right\}.$$

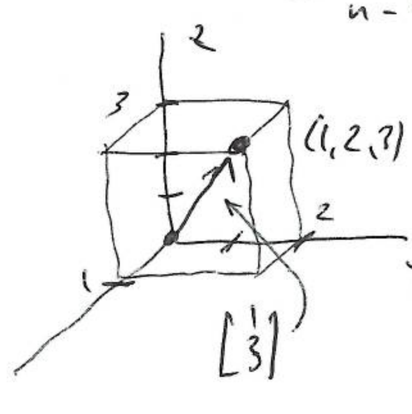


where $\begin{bmatrix} x \\ y \end{bmatrix}$ is the vector, based at $(0,0)$ and with head at $(x,y) \in \mathbb{R}^2$.

We can generalize this readily to $\mathbb{R}^n$:

$$\mathbb{R}^n = \{ (x_1, x_2, \cdots, x_n) \mid x_i \in \mathbb{R},\ i = 1, \cdots, n \}$$

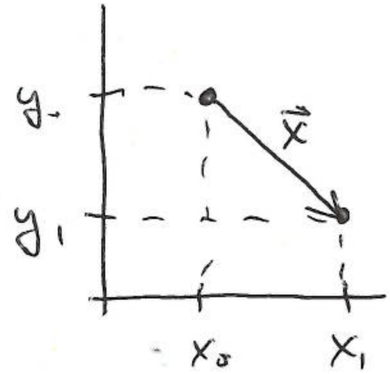
$$= \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \;\middle|\; x_i \in \mathbb{R},\ i = 1, \cdots, n \right\} = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{-times}}$$



For $n=3$, this is easy to visualize

For $n>3$, this is not. But we can still talk about and study n-vectors and $\mathbb{R}^n$.

We can also talk about vectors in $\mathbb{R}^2$ based at $(x_0, y_0) \neq (0,0)$ with head at (x_1, y_1). This vector has components $\vec{x} = \begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}$.

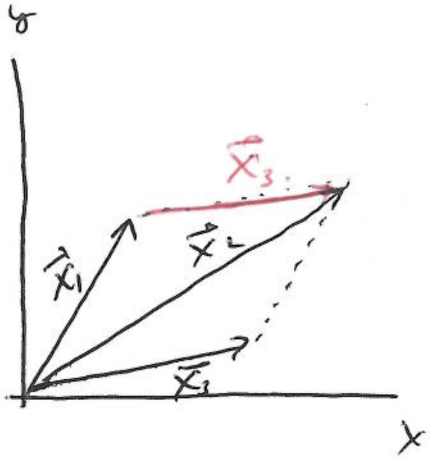


ex: The vector $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ based at $(1,2)$ has head at $(2,1)$.

"Free" vectors like this also generalize to $\mathbb{R}^n$ readily and are useful for many reasons:

1. Vector fields are crucial in mathematical modeling. We will learn and use them in this class.

2. Points can be added in $\mathbb{R}^2$ (and $\mathbb{R}^n$), component-wise, so thus can vectors, and geometrically, free vectors help visualize vector addition in $\mathbb{R}^n$.



Here $\vec{x}_2 = \vec{x}_1 + \vec{x}_3$, so also $\vec{x}_3 = \vec{x}_2 - \vec{x}_1$, where we can interpret $\vec{x}_3$ as the vector based at the head of $\vec{x}_1$ and with its head at the head of $\vec{x}_2$

The operation of addition on pts and vectors $(x_1, \cdots, x_n)$, and $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ holds even when we cannot geometrically envision them.

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

and we can always multiply a pt or a vector by a scalar:

$$c \cdot \vec{x} = c \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c x_1 \\ \vdots \\ c x_n \end{bmatrix}, \quad c \in \mathbb{R},\ \vec{x} \in \mathbb{R}^n$$

But we cannot generally define a multiplication of vectors that makes sense in $\mathbb{R}^n$.

There is a common notion of multiplication in $\mathbb{R}^n$, but the product is not a vector. It is a scalar:

$$\vec{x}\cdot\vec{y} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}\cdot\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix} = x_1y_1+\cdots+x_ny_n = \sum_{i=1}^{n} x_iy_i$$

dot product

Note: A matrix is simply an array of numbers

$$A_{m\times n} = \begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & & \\ a_{m1} & \cdots & a_{mn}\end{bmatrix}.$$

Given any such A, there is another array called the transpose of A, A^T, whose each element a_{ij} of A corr to a_{ji} of A^T.

Switch rows and columns.

ex: $A = \begin{bmatrix}2 & 3 & 5\\ 1 & 4 & 6\end{bmatrix}$, then $A^T = \begin{bmatrix}2 & 1\\ 3 & 4\\ 5 & 6\end{bmatrix}$.

ex: $B = \begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}$, $B^T = \begin{bmatrix}1 & 3\\ 2 & 5\end{bmatrix}$.

ex: $\vec{x} = \begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix}$. $\vec{x}^T = \begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$.

Then, the dot product is

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n$$

dot prod.

$$= [x_1 \cdots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{x}^T y.$$

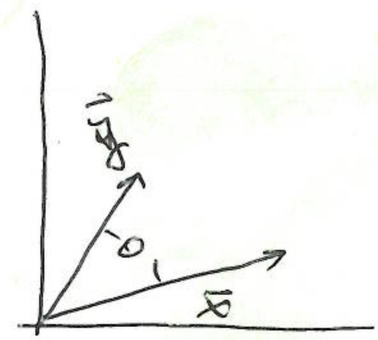
standard matrix mult.

This notion of a scalar product in $\mathbb{R}^n$ is vital for visualizing things in $\mathbb{R}^n$ (doing geometry).

In $\mathbb{R}^2$, any two vectors form an angle between them.

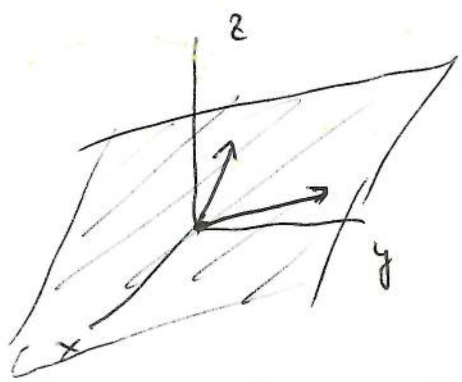
where

$$\theta = \cos^{-1}\left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}\right).$$



and where $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$ is called the length of the vector $\vec{x}$.

This also works in $\mathbb{R}^n$, $n > 2$, since for any 2 non collinear vectors in $\mathbb{R}^n$, they will always form a plane. Thus θ will be well-defined



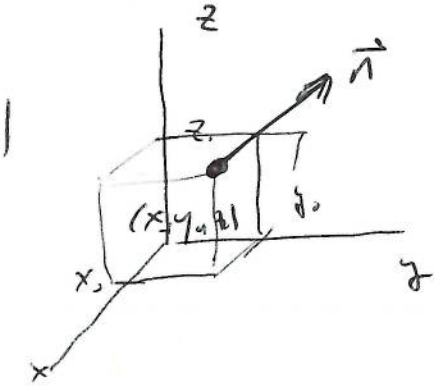
① Note that $\cos^{-1}(0) = \frac{\pi}{2}$:
2 vectors are perpendicular when $\vec{x} \cdot \vec{y} = 0$.

② Planes in $\mathbb{R}^3$, like lines in $\mathbb{R}^2$, are important objects in vector calculus.

The dot product can also help to visualize a plane in $\mathbb{R}^3$: Let $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$

$$= \begin{bmatrix} A \\ B \\ C \end{bmatrix}.$$

be a vector in $\mathbb{R}^3$ based at (x_0, y_0, z_0).



Any other vector $\vec{v}$ based at (x_0, y_0, z_0) with head at (x, y, z) will have components $\vec{v} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$

If $\vec{v}$ is chosen to be perpendicular to $\vec{n}$, then

$$\vec{n} \cdot \vec{v} = 0 = A(x - x_0) + B(y - y_0) + C(z - z_0)$$

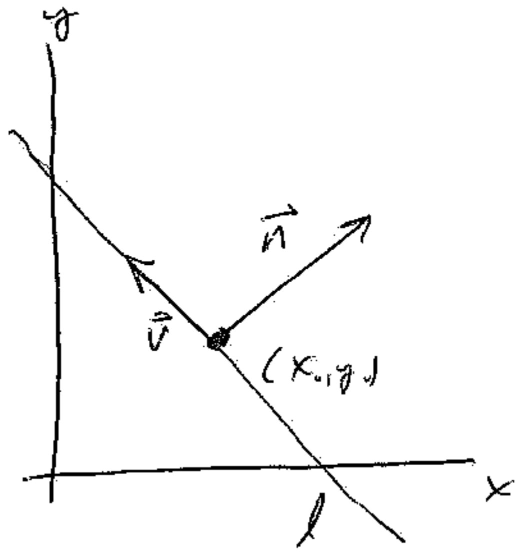
or $\boxed{Ax + By + Cz + D = 0}$ (where $D = \underbrace{-Ax_0 - By_0 - Cz_0}_{\text{(just a \#)}}$)

This is just the standard form for the eqn of a plane in $\mathbb{R}^3$: For any ~~chosen~~ pt (x_0, y_0, z_0) and any choice of $A, B, C \in \mathbb{R}$ (not all 0), the set of all solutions $(x, y, z) \in \mathbb{R}^3$ is a plane perpendicular to $\vec{n} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ based at (x_0, y_0, z_0).

You have seen this before:

The standard eqn for a line in $\mathbb{R}^2$ is

$$Ax + By + C = 0, \quad A, B, C \in \mathbb{R}.$$



Given any $\vec{n} = A\vec{i} + B\vec{j} = \begin{bmatrix} A \\ B \end{bmatrix}$

based at $(x_0, y_0) \in \mathbb{R}^2$

any vector $\vec{v} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$.

based at (x_0, y_0) is perp.

to $\vec{n}$ if $\vec{n} \circ \vec{v} = 0$

$$= \begin{bmatrix} A \\ B \end{bmatrix} \circ \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} = A(x - x_0) + B(y - y_0)$$

$$= Ax + By + C = 0$$

All of these vectors $\vec{v}$ form a line (where $C = \underbrace{-Ax_0 - By_0}_{\text{just a \#}}$.

The eqn for l is $Ax + By + C = 0$.

Do the same in $\mathbb{R}^4$: $\vec{n} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$ $\vec{v} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \\ u - u_0 \end{bmatrix}$ based at $(x_0, y_0, z_0, u_0) \in \mathbb{R}^4$, then

$\vec{n} \cdot \vec{v} = 0 \iff Ax + By + Cz + Du + E = 0$ is the eqn of what kind of space? A 3-dim subspace of $\mathbb{R}^4$: $u = \frac{-E - Ax - By - Cz}{D}$ (if $D \neq 0$)

Note: An $(n-1)$-dim linear space living inside an n-dim space is called a <u>hyperplane</u>

ex. A line in $\mathbb{R}^2$, a plane in $\mathbb{R}^3$, ...

New Q: If $Ax + By + Cz + D = 0$ describes a plane in $\mathbb{R}^3$, then what eqn describes a line in $\mathbb{R}^3$?

Hint: There isn't one!