TANGENT SPACES.

110.202 CALCULUS III
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Here is an odd thought that may make sense after some reflection: When studying a function like $g\colon \mathbb{R}^2 \to \mathbb{R}$, we can form the graph in $\mathbb{R}^3$ as the set of all solutions to the equation $z = g(x, y)$. Then, if the function is differentiable at a point (x_0, y_0), we can form the tangent space to the graph of g at the point $(x_0, y_0, z_0) = (x_0, y_0, g(x_0, y_0))$, and we get the equation

$$
\begin{aligned}
z &= z_0 + \frac{\partial g}{\partial x}(x_0, y_0)\,(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)\,(y - y_0) \qquad (1)\\
&= g(x_0, y_0) + \begin{bmatrix} \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \\
&= g(x_0, y_0) + Dg(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \\
&= g(x_0, y_0) + \nabla g(x_0, y_0) \bullet \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix},
\end{aligned}
$$

where $\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$ is a 2-vector based at $(x_0, y_0) \in \mathbb{R}^2$. This was detailed on page 110 of the text. This is the equation of a plane in $\mathbb{R}^3$, and written in terms of a (linear) function z as a function of x and y. It is *the best linear approximation to the function* $g(x, y)$ *at the point* (x_0, y_0).

More recently, we learned the following: If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is differentiable at a point (x_0, y_0, z_0), and if $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$, then we can write the tangent space to the c-level set

$$
\mathcal{S}_c = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; f(x, y, z) = f(x_0, y_0, z_0) = c \right\}
$$

as the set of all vectors in $\mathbb{R}^3$ that are perpendicular to the gradient of f at (x_0, y_0, z_0); vectors $\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$, based at (x_0, y_0, z_0) that are tangent to $\mathcal{S}_c$ will satisfy

$$
\begin{aligned}
0 = \nabla f(x_0, y_0, z_0) \bullet \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} &= \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0, z_0) \\ \frac{\partial f}{\partial y}(x_0, y_0, z_0) \\ \frac{\partial f}{\partial z}(x_0, y_0, z_0) \end{bmatrix} \bullet \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \\
&= \frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0). \qquad (2)
\end{aligned}
$$

How are these concepts related? If we compare these two, we immediately see that they both describe subsets of $\mathbb{R}^3$, one concerning the graph of an $\mathbb{R}^2$-function, and the other as a level set of an $\mathbb{R}^3$-function.

To start, let's compare the two equations directly:

$$0 = \frac{\partial g}{\partial x}(x_0, y_0)\,(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)\,(y - y_0) - (z - z_0) \tag{1}$$

$$0 = \frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0). \tag{2}$$

Consider these two together under the guise that $f(x, y, z) = g(x, y) - z$. Then, the graph of g, seen as the set of solutions to the equation in $\mathbb{R}^3$ given by $z = g(x, y)$, can be viewed as the 0-level set of f:

$$\mathcal{S}_0 = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; f(x, y, z) = g(x, y) - z = 0 \right\}.$$

Let use this and redo the calculation: We have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[g(x, y) - z \right] = \frac{\partial g}{\partial x} \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[g(x, y) - z \right] = \frac{\partial g}{\partial y} \\
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[g(x, y) - z \right] = -1.
\end{aligned}$$

They are the same equation, reflecting the fact that the tangent space is always defined the same way. It is just that the interpretation of the set of solutions, either that of $f(x, y, z) = 0$, or $z = g(x, y)$ requires a slightly different language to define each.

I hope that you found this useful.