THE VOLUME ENCLOSED BY A SPHERE OF RADIUS $r \geq 0$ IN $\mathbb{R}^3$.

110.202 CALCULUS III
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One learns early that there are general formulae for measuring certain quantities inherent to geometric objects. Things like the area of a triangle or the perimeter of a pentagon can be derived purely using geometric means. However, calculus also allows for effective calculation, and, of course, the formulae must agree. Here we calculate the volume of $\mathcal{S}_r^2$, a 2-dimensional sphere of radius $r \geq 0$, in three space. Geometrically, it is known that

$$\textbf{volume}(\mathcal{S}_r^2) = \frac{4}{3}\pi r^3.$$

Using calculus, we reestablish this formula.

Exercise. Calculate, using calculus, the volume enclosed by a $\mathcal{S}_r^2 \subset \mathbb{R}^3$.

Note that where $\mathcal{S}_r^2$ is located in $\mathbb{R}^3$ does not matter, so we simply place it centered at the origin of the xyz-space $\mathbb{R}^3$. We will do this in a number of ways:

The interior of $\mathcal{S}_r^2$ viewed as an elementary solid region $\mathcal{W}_r \subset \mathbb{R}^3$:

Noting that the r-sphere $\mathcal{S}_r$ encloses a solid $\mathcal{W}_r$ which is elementary, we can use Example 3 of Section 5.5 directly to write

$$\mathcal{W}_r = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} -r \leq x \leq r, \\ -\sqrt{r^2 - x^2} \leq y \leq \sqrt{r^2-x^2}, \\ -\sqrt{r^2-x^2-y^2} \leq z \leq \sqrt{r^2-x^2-y^2} \end{array} \right\}.$$

Then, integrating the function $f(x,y,z) = 1$ over $\mathcal{W}_r$ will yield the volume of $\mathcal{W}_r$. Thus

$$\textbf{volume}(\mathcal{W}_r) = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} 1\, dz\, dy\, dx. \tag{1}$$

The calculation is now to systematically calculate this triple integral by viewing the integral as a nested integral and integrating from the inside out.

Firstly, we have

$$
\begin{aligned}
\textbf{volume}(\mathcal{W}_r) &= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} 1\,dz\,dy\,dx \\
&= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left(\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} 1\,dz\right) dy\,dx \\
&= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \left(z\Big|_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}}\right) dy\,dx \\
&= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 2\sqrt{r^2-x^2-y^2}\,dy\,dx. \qquad (2)
\end{aligned}
$$

Next, we focus on the new "inside" integral, where

$$
\begin{aligned}
\textbf{volume}(\mathcal{W}_r) &= \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 2\sqrt{r^2-x^2-y^2}\,dy\,dx \\
&= \int_{-r}^{r}\left(\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 2\sqrt{r^2-x^2-y^2}\,dy\right) dx.
\end{aligned}
$$

To make this inside integral easier to work with, we note that the part $r^2 - x^2$ is a constant when integrating only over y. So for a minute, let's write this constant as $a^2 = r^2 - x^2$, for $a \geq 0$. (Why is this okay?) Then

$$\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 2\sqrt{r^2-x^2-y^2}\,dy = \int_{-a}^{a} 2\sqrt{a^2-y^2}\,dy.$$

From Calculus I, one way to solve this integral is with the trigonometric substitution $y = a\sin t$. Then, along with $dy = a\cos t\,dt$ (and the limits: when $y = -a$, $t = -\frac{\pi}{2}$, and when $y = a$, $t = \frac{\pi}{2}$ on an injective interval), we have

$$
\begin{aligned}
\int_{-a}^{a} 2\sqrt{a^2-y^2}\,dy &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{a^2-a^2\sin^2 t}\,a\cos t\,dt \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\cos^2 t\,dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^2\left(1+\cos 2t\right)\,dt \\
&= a^2\left(t + \frac{1}{2}\sin 2t\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi a^2 = \pi\left(r^2-x^2\right).
\end{aligned}
$$

Hence

$$\textbf{volume}(\mathcal{W}_r) = \int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 2\sqrt{r^2-x^2-y^2}\,dy\,dx = \int_{-r}^{r} \pi\left(r^2-x^2\right)\,dx.$$

Now to finish, using standard first semester calculus ideas,

$$\begin{aligned}\textbf{volume}(\mathcal{W}_r) &= \int_{-r}^{r} \pi\left(r^2 - x^2\right)\, dx = \pi\left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} \\ &= \pi\left(r^2 r - \frac{r^3}{3} - r^2(-r) + \frac{(-r)^3}{3}\right) = \pi\left(2r^3 - \frac{2}{3}r^3\right) = \frac{4}{3}\pi r^3.\end{aligned}$$

The sphere $\mathcal{S}_r^2$ viewed as the difference between two functions over a planar domain:

Noting that the r-sphere is defined as

$$\mathcal{S}_r = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 = r^2\right\},$$

we can write the upper hemisphere as the function $z = f(x,y) = \sqrt{r^2 - x^2 - y^2}$, and the lower hemisphere as the function $z = g(x,y) = -\sqrt{r^2 - x^2 - y^2}$, both of which are defined and continuous on the domain

$$\mathcal{D}_r = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 \leq r^2\right\}.$$

Then the solid region between the two functions is enclosed precisely by $\mathcal{S}_r$. The volume of this region is then

$$\textbf{volume}(\mathcal{S}_r) = \iint_{\mathcal{D}} \left(f(x,y) - g(x,y)\right)\, dA = \iint_{\mathcal{D}} 2\sqrt{r^2 - x^2 - y^2}\, dA. \tag{3}$$

To solve this integral, we note that $\mathcal{D}$ is already parameterized by x and y, and that $\mathcal{D}$ is elementary of Type III in the plane. So considering it a Type I region in the plane, we can write

$$\mathcal{D}_r = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} -r \leq x \leq r, \\ -\sqrt{r^2 - x^2} \leq y \leq \sqrt{r^2 - x^2} \end{array}\right\}.$$

Then the integral becomes

$$\textbf{volume}(\mathcal{S}_r) = \iint_{\mathcal{D}} 2\sqrt{r^2 - x^2 - y^2}\, dA = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 2\sqrt{r^2 - x^2 - y^2}\, dy\, dx.$$

This last expression is precisely the one above in Equation 2. The rest of the calculation follows, as above, from there.

Changing variables to spherical coordinates in $\mathbb{R}^3$

Another way to calculate volume is to switch coordinate systems in $\mathbb{R}^3$ form the rectilinear (x,y,z) to the spherical (ρ,θ,φ). One reason for this is that, in spherical coordinates, the sphere $\mathcal{S}_r^2$ and its interior (that is, the space $\mathcal{W}_r$) is just a cuboid (a 3-dimensional version of a rectangle). Indeed, consider the set of transformations

$$\begin{aligned}T\left(\rho,\theta,\varphi\right) &= \left(x(\rho,\theta,\varphi), y(\rho,\theta,\varphi), z(\rho,\theta,\varphi)\right) \\ &= \left(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi\right),\end{aligned}$$

which takes the cuboid

$$\mathcal{W}_r^* = [0,r] \times [0,2\pi] \times [0,\pi]$$

to $\mathcal{W}_r$, as seen in Figure 1 below.

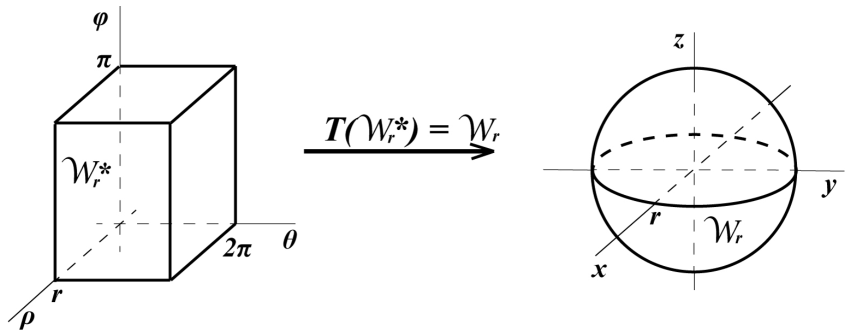

FIGURE 1. The transformation from rectilinear coordinates to spherical coordinates.

We can integrate in this new coordinate system as long as we follow the Change of Variables Theorem and include the absolute value of the Jacobian determinant of the transformation, so

$$\iiint_{\mathcal{W}_r} dx\,dy\,dz = \iiint_{\mathcal{W}_r^*} \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} \right| d\rho\,d\theta\,d\varphi.$$

One first step in this process is to calculate the Jacobian (determinant of the transformation):

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \\
&= \cos\varphi \begin{vmatrix} -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \end{vmatrix} + (-\rho\sin\varphi) \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi \end{vmatrix} \\
&= -\rho^2\cos^2\varphi\left(\sin^2\theta\sin\varphi + \cos^2\theta\sin\varphi\right) - \rho^2\sin^2\varphi\left(\cos^2\theta\sin\varphi + \sin^2\theta\sin\varphi\right) \\
&= -\rho^2\left(\sin^2\theta\sin\varphi + \cos^2\theta\sin\varphi\right) = -\rho^2\sin\varphi.
\end{aligned}$$

At this point, then, we can also realize that $\rho \in [0, r]$ is nonnegative, and on the interval $\varphi \in [0, \pi]$, so is $\sin\varphi$. Hence

$$\left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} \right| = \left| -\rho^2 \sin\varphi \right| = \rho^2 \sin\varphi.$$

Thus

$$
\begin{aligned}
\textbf{volume}\,(\mathcal{W}_r) &= \iiint_{\mathcal{W}_r} dx\,dy\,dz = \iiint_{\mathcal{W}_r^*} \left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} \right| d\rho\,d\theta\,d\varphi \\
&= \int_0^r \int_0^\pi \int_0^{2\pi} \rho^2 \sin\varphi\,d\theta\,d\varphi\,d\rho \\
&= \int_0^r \int_0^\pi \left(\theta\rho^2 \sin\varphi \Big|_0^{2\pi} \right) d\varphi\,d\rho = 2\pi \int_0^r \int_0^\pi \rho^2 \sin\varphi\,d\varphi\,d\rho \\
&= 2\pi \int_0^r \left(-\rho^2 \cos\varphi \Big|_0^{\pi} \right) d\rho = 2\pi \int_0^r \left(-\rho^2 \cos\pi + \rho^2 \cos 0 \right) d\rho \\
&= 4\pi \int_0^r \rho^2\,d\rho = 4\pi \left(\frac{\rho^3}{3} \Big|_0^r \right) = \frac{4}{3}\pi r^2,
\end{aligned}
$$

as before.