CALCULUS III
FALL 2019
PROBLEM SET 9 EXERCISE 3 SOLUTION

Since g is continuous on $[c,d]$ and f is continuous on $[a,b]$, we have $\tilde{g}(x,y) = g(y)$ and $\tilde{f}(x,y) = f(x)$ are continuous on $R = [a,b] \times [c,d]$. Therefore the product $h(x,y) = \tilde{g}(x,y)\tilde{f}(x,y) = f(x)g(y)$ is a continuous function on R. Since h is continuous, we may apply Fubini's theorem to say

$$\int\int_R h(x,y)dA = \int_a^b \int_c^d h(x,y)dydx.$$

So we can now calculate

$$\begin{aligned}
\int\int_R f(x)g(y)dA &= \int_a^b \left[\int_c^d f(x)g(y)dy\right]dx \\
&= \int_a^b f(x)\left[\int_c^d g(y)dy\right]dx \\
&= \left[\int_c^d g(y)dy\right]\int_a^b f(x)dx \\
&= \left[\int_a^b f(x)dx\right]\left[\int_c^d g(y)dy\right].
\end{aligned}$$

In the second line, we pulled $f(x)$ out of the inner integral since we are integrating with respect to y, and therefore treating $f(x)$ as a constant. To obtain the third line, notice that $\int_c^d g(y)dy$ is equal to some constant (since it is a definite integral) and similarily we can pull it out in front of the dx integral.

Some students used the Fundamental Theorem of Calculus to show the result (by calculating integrals explicitly with antiderivatives). This works but is not necessary.