EXAMPLE: ARCLENGTH

AS.110.202 CALCULUS III
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Problem (**4.2.8**)**.** A rolling circle of radius R traces out a cycloid, which can be parameterized by $\mathbf{c}(t) = \begin{bmatrix} R(t-\sin t) \\ R(1-\cos t) \end{bmatrix}$. One arch of the curve runs from $t = 0$ to $t = 2\pi$. Show that the length of this single arch is always four times the diameter of the circle.

Strategy. We integrate the speed of the curve given by the parameterization, on the interval $[0, 2\pi]$.

Solution. For the parameterization given , the velocity is then

$$\mathbf{c}'(t) = \begin{bmatrix} \frac{d}{dt}\left[R(t-\sin t)\right] \\ \frac{d}{dt}\left[R(1-\cos t)\right] \end{bmatrix} = \begin{bmatrix} R(1-\cos t) \\ R\sin t) \end{bmatrix},$$

so that the speed at time t is

$$\begin{aligned} ||\mathbf{c}'(t)|| &= \sqrt{(R(1-\cos t))^2 + (R\sin t)^2} \\ &= \sqrt{R^2 - 2R^2\cos t + R^2\cos^2 t + R^2\sin^2 t} \\ &= \sqrt{R^2 - 2R^2\cos t + R^2(\cos^2 t + \sin^2 t)} \\ &= R\sqrt{2 - 2\cos t}. \end{aligned}$$

Now, we calculate the arclength

$$L(\mathbf{c}) = \int_0^{2\pi} ||\mathbf{c}'(t)||\, dt = R\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt.$$

To work with this integral, we first use the identity

$$2 - 2\cos t = 4\left(\frac{1-\cos t}{2}\right) = 4\sin^2\left(\frac{t}{2}\right).$$

Then

$$\begin{aligned} L(\mathbf{c}) &= R\int_0^{2\pi} \sqrt{2-2\cos t}\, dt \\ &= R\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{t}{2}\right)}\, dt = 2R\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\, dt. \end{aligned}$$

We note here that, normally, we would need the absolute value signs around the integrand when we take the square root. Bit on the interval $[0, 2\pi]$, the function $\sin\left(\frac{t}{2}\right) \geq 0$. Hence we do not need them. Hence we can finish the calculation now:

$$\begin{aligned} L(\mathbf{c}) &= 2R\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\, dt = 2R\left(-2\cos\left(\frac{t}{2}\right)\right)\Bigg|_0^{2\pi} \\ &= 4R\left(-\cos\pi + \cos 0\right) = 8R. \end{aligned}$$

And finally, $8R = 4D$, where D is the diameter of the circle. THis completes the exercise.