EXAMPLE: CONSTRAINED EXTREMA

AS.110.202 CALCULUS III
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Problem (3.4.6). Find the extrema of $f(x, y, z) = x + y + z$ subject to the constraints $x^2 - y^2 = 1$ and $2x + z = 1$.

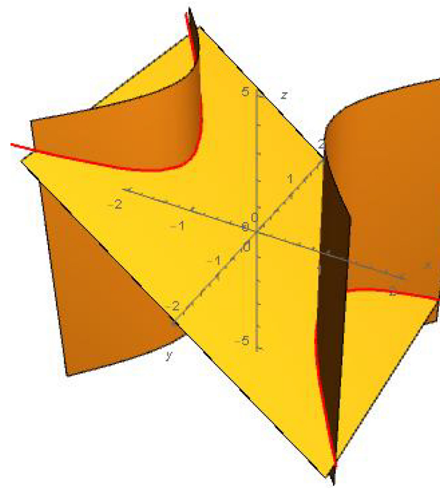

FIGURE 1. The constraint space is a curve in $\mathbb{R}^3$.

Strategy. We use the method of Lagrange Multipliers with multiple constraints to find the critical points of f along the space $\mathcal{S}$ defined by the two constraints. Then we evaluate the critical points to choose the extrema of f, if they exist.

Solution. For the Method of Lagrange Multipliers, we need to define the constraints as level sets of functions on the domain of f. So define $g_1(x, y, x) = x^2 - y^2 - 1$, and $g_2(x, y, z) = 2x - z$. Then the space

$$\mathcal{S} = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, g_1(x, y, z) = g(2(x, y, z) = 0 \right\}$$

comprises the intersection of these two constraints, and is a curve (See the red curve in Figure 1. Note that the 0-level set of g_1 is a two-sheeted hyperboloid symmetric about the x-axis and the yz-plane, while the 0-level set of g_2 is a plane, as shown in Figure 1.

By the ideas behind the Method of Lagrange Multipliers, we know that if there are extrema of $f\big|_{\mathcal{S}}$, they will occur at places where

$$\nabla f(x, y, z) = \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z).$$

But since $\mathcal{S}$ is a closed, but not bounded set, $f\big|_{\mathcal{S}}$ may or may not achieve its absolute max and absolute min. If there are extrema, they will occur at the critical points of $f\big|_{\mathcal{S}}$. So, to find possible critical points of $f\big|_{\mathcal{S}}$, we solve the system

$$\begin{aligned}
\nabla f(x, y, z) &= \lambda_1 \nabla g_1(x, y, z) + \lambda_2 \nabla g_2(x, y, z) \\
g_1(x, y, z) &= 0 \\
g_2(x, y, z) &= 0.
\end{aligned}$$

This is a system of 5 equations in 5 unknowns:

$$\begin{aligned}
1 &= 2\lambda_1 x + 2\lambda_2 \\
1 &= -2\lambda_1 y \\
1 &= \lambda_2 \\
x^2 - y^2 &= 1 \\
2x + z &= 0.
\end{aligned}$$

Immediately, we see that $\lambda_2 = 1$, and so so the top two equations become $1 = -2\lambda_1 x$ and $1 = -2\lambda_1 y$. Solving these two equations, for x and y in each, results in the conclusion that $x = y$. But this constraint is incompatible with the constraint $x^2 - y^2 = 1$. Hence the conclusion is that there are no critical points of $f\big|_{\mathcal{S}}$. Again, this is possible, since $\mathcal{S}$ is not a bounded set. Hence there are no extrema of $f\big|_{\mathcal{S}}$. This completes the problem.