

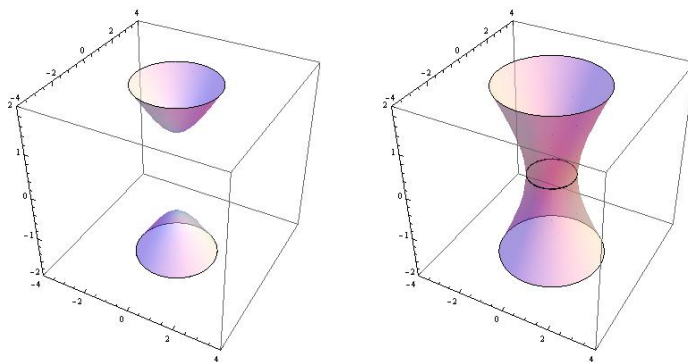
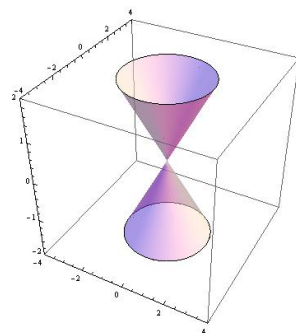
EXAMPLE: LEVEL SETS OF A FUNCTION ON $\mathbb{R}^3$

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The set of solutions in $\mathbb{R}^3$ to the equation given by $z^2 = x^2 + y^2$ is called the *elliptic cone*, whose graph is given at right. Recognize that this is NOT the graph of a function $z = f(x, y)$, and even if we were to take ONLY the positive root when “solving” for z , the resulting graph would be different from that of the paraboloid, the function on two variables corresponding to the equation $z = x^2 + y^2$.

If we change the equation slightly by adding a constant to the right-hand side, the graph changes dramatically:

- The set $\{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2 - 1\}$ is an example of a *one-sheeted hyperboloid* (are you also thinking of the cooling tower of a nuclear power plant?), while
- the set $\{(x, y, z) \in \mathbb{R}^3 \mid z^2 = x^2 + y^2 + 1\}$ is called a *two-sheeted hyperboloid*.



Now consider the scalar-valued function

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad f(x, y, z) = x^2 + y^2 - z^2.$$

The graph of f is best “seen” as a 3-dimensional hypersurface (space of size one-dimension less than the ambient space) of $\mathbb{R}^4$. Since we cannot, therefore, view it directly, we can study this function via its level sets. These turn out to be surfaces living in the domain $\mathbb{R}^3$. What do they look like?

For $c \in \mathbb{R}$, the set

$$f^{-1}(c) = \{(x, y, z) \in \mathbb{R}^3 \mid c = x^2 + y^2 - z^2\},$$

or the solution set to the equation $z^2 = x^2 + y^2 - c$. Sets of this type are graphed above, no? We have

- $f^{-1}(0) =$ elliptic cone.

- $f^{-1}(0)|_{c < 0} = 2\text{-sheeted hyperboloid, and}$
- $f^{-1}(0)|_{c > 0} = 1\text{-sheeted hyperboloid.}$

Below are the graphs of a few of these level sets to give you a picture of how they look together.

