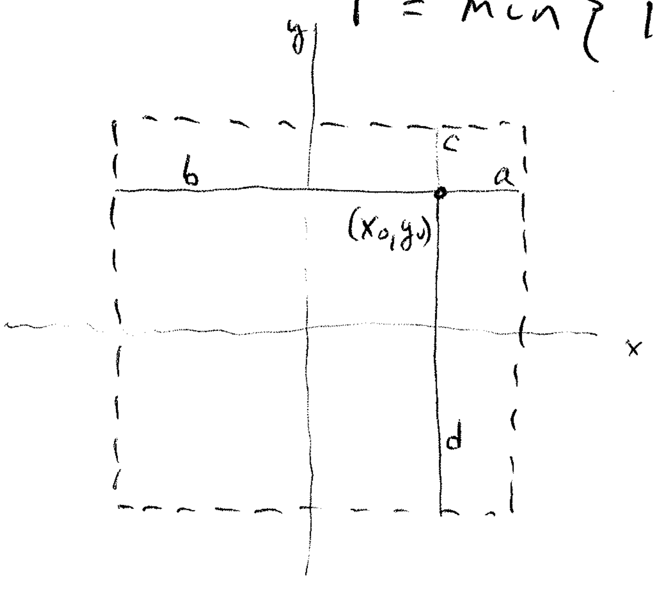
Problem 2.2.18

Let $A = \{(x,y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < 1\}$. Show that A is an open subset of $\mathbb{R}^2$.

Strategy: We need to show that for any pt $(x_0, y_0) \in A$, there is an $\varepsilon > 0$ where $B_\varepsilon(x_0, y_0)$ lies completely inside A. We do this by construction.

Solution: Choose any $(x_0, y_0) \in A$, so that $-1 < x_0 < 1$, $-1 < y_0 < 1$. Define

$$r = \min\{1 - x_0,\ x_0 - (-1),\ 1 - y_0,\ y_0 - (-1)\}.$$



These are the 4 distances along lines parallel to the axes between (x_0, y_0) and the 4 boundary lines of A, labeled a, b, c, d, respectively.

and r is defined here as the smallest of these 4 distances.

Note 2 things:

① The distance between any pt in A and the boundary of A will always be ~~almost~~ at the minimum of the 4 distances along those directions parallel to the axes. The triangle inequality is one way to see this. There are others. (see below)

② This is also true for any open ball of radius less than r centred at (x_0, y_0), that its distance to the boundary of A will be positive and the minimum of the distances measured in the vertical and horizontal directions.

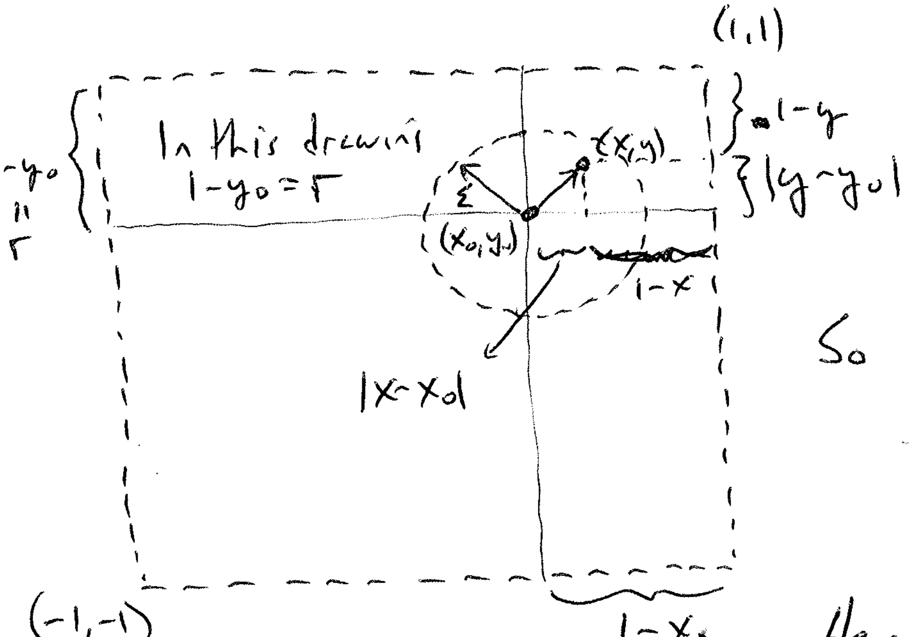
You can actually prove this using techniques in SVC. I won't ask you to, though.

Now choose $\varepsilon = \frac{1}{2}r$ (you can choose any $\varepsilon > 0$ which is less than or equal to r here. I simply choose this one), and construct

$$B_\varepsilon(x_0, y_0) = \{(x,y) \in \mathbb{R}^2 \mid d((x,y) - (x_0,y_0)) < \varepsilon\}.$$

Claim: Every point $(x,y) \in B_\varepsilon(x_0,y_0)$ is in A. when $\varepsilon = \frac{1}{2}r$. To see this, we show that for any pt $(x,y) \in B_\varepsilon(x_0,y_0)$,

$$-1 < x < 1, \quad -1 < y < 1.$$



Here there are 4 calculations:

① We know $\underbrace{1-x_0}_{\text{distance to right edge.}} \geq r$.

So $r \leq 1 - x_0 = 1 - x + x - x_0$
$\leq 1 - x + |x - x_0|$
(since $a + b \leq a + |b|$)
$< 1 - x + \frac{1}{2}r$

Here $r < 1 - x + \frac{1}{2}r$, or $\frac{1}{2}r < 1 - x$ and since $\frac{1}{2}r$ is positive $0 < 1 - x$ so that $\boxed{x < 1}$

(2) We also know $r \leq x_0 - (-1) = x_0 + 1$, where $x_0 + 1$ is the distance to the ~~right end~~ left edge of the box. Hence

$$r \leq x_0 - (-1) = x_0 + 1 = x_0 - x + x + 1 \leq |x_0 - x| + x + 1$$

(same reason as before)

$$< \tfrac{1}{2}r + x + 1$$

(since $|x_0 - x| < \varepsilon = \tfrac{1}{2}r$)

But then $r < \tfrac{1}{2}r + x + 1$, or $\tfrac{1}{2}r < x + 1$ and since $\tfrac{1}{2}r$ is positive, $0 < x + 1$, or $\boxed{-1 < x}$

Hence by (1) and (2) $-1 < x < 1$.

(3) and (4) are just like (1) and (2), so that

$-1 < y < 1$.

Thus if $(x,y) \in B_\varepsilon(x_0, y_0)$, where $\varepsilon = \tfrac{1}{2}r$, then $(x,y) \in A$.

Thus $B_\varepsilon(x_0,y_0) \subset A$ and we are done. ∎