# Problem 1.2.26

Find the line through (3, 1, −2) that intersects and is perpendicular to the line

$$\ell: \; x = -1 + t, \; y = -2 + t, \; z = -1 + t.$$

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**Note:** There are a number of ways to solve this problem. I will employ 2 of them here.

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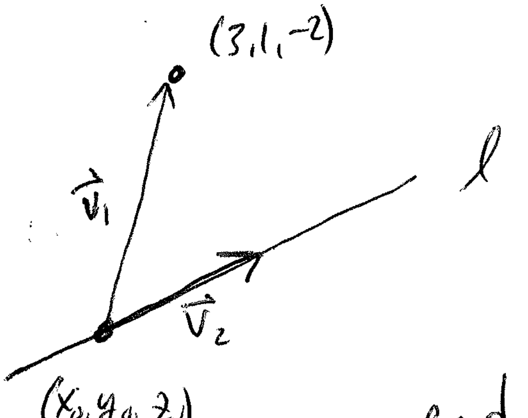
Method 1: Using the dot product directly.

Strategy: Choose an arbitrary pt $(x_0, y_0, z_0)$ on the line $\ell$ and construct 2 vectors: One, $\vec{v}_1$, along the line, and $\vec{v}_2$, from $(x_0, y_0, z_0)$ to $(3, 1, -2)$. We then take the dot product of $\vec{v}_1$ and $\vec{v}_2$ and solve for $(x_0, y_0, z_0)$ where these 2 vectors are perpendicular (dot product is 0).



Strategy (cont'd) We then construct the equations for the line through this choice of $(x_0, y_0, z_0)$ and $(3, 1, -2)$.

Solution: Using the graph above, the vector $\vec{v}_1 = \begin{bmatrix} 3 - x_0 \\ 1 - y_0 \\ -2 - z_0 \end{bmatrix}$. We construct $\vec{v}_2$ knowing that after 1 unit of time, the base of $\vec{v}_2$ is $(x_0, y_0, z_0) = (-1 + t_0, -2 + t_0, -1 + t_0)$ for some $t_0$, and the head is $(x, y, z) = (-1 + (t_0+1), -2 + (t+1), -1 + (t+1))$ so that $\vec{v}_1 = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = \begin{bmatrix} -1 + (t+1) - (-1+t) \\ -2 + (t+1) - (-2+t) \\ -1 + (t+1) - (-1+t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

The choice of $(x_0, y_0, z_0)$ where $\vec{v}_1 \cdot \vec{v}_2 = 0$ is

$$\vec{v}_1 \cdot \vec{v}_2 = 0 = \begin{bmatrix} 3 - x_0 \\ 1 - y_0 \\ -2 - z_0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3 - x_0 + 1 - y_0 - 2 - z_0 = 0$$

And since $(x_0, y_0, z_0)$ is on the line $\ell$, we set

$$3 - (-1+t) + 1 - (-2+t) - 2 - (-1+t) = 0 = 2 + 4 - 3t = 0$$

Here on $l$, where $t=2$, $\vec{v}_1$ and $\vec{v}_2$ are perp.

This is at the point

$$(x_0, y_0, z_0) = (-1+(2), -2+(2), -1+(2)) = (1,0,1).$$

This is the pt of intersection of $l$ and the line through $(1,0,1)$ and $(3,1,-2)$ are perpendicular.

The equations of this line are

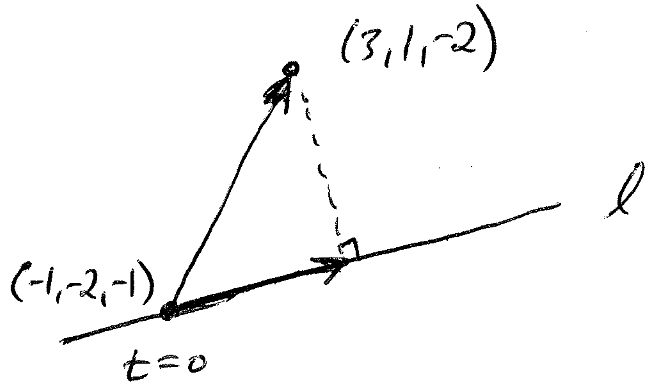
$$\cancel{x_0} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \left( \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) t$$

or

$$\boxed{\begin{aligned} x &= 1+2t \\ y &= t \\ z &= 1-3t \end{aligned}}$$

Check: Are $\vec{v}_1 = \begin{bmatrix} 3-1 \\ 1-0 \\ -2-1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ perpendicular?

Yes since $\vec{v}_1 \cdot \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2(1) + (1)1 + (-3)(1) = 0.$




## Method 2

**Strategy:**

We form a vector $\vec{v}_2$ from the pt $t=0$ on the line $l$ and $(3,1,-2)$. Then we take the orthogonal projection of $\vec{v}_2$ onto the line $l$. The head of the orthogonal projection is the pt $(x_0,y_0,z_0)$ where the line through $(x_0,y_0,z_0)$ and $(3,1,-2)$ ~~are~~ is perpendicular to $l$. (By definition of orthogonal projection).

**Solution** The previous solution detailed how to construct a vector along $l$, $\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$.

Call the orthogonal projection of $\vec{v}_2$ onto $\vec{v}_1$ $\vec{v}_p$.

At $t=0$ on $l_1$ is the pt $(x,y,z)=(-1,-2,-1)$.
Hence the vector $\vec{v}_2 = \begin{bmatrix} 3-(-1) \\ 1-(-2) \\ -2-(-1) \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$ (see drawing above)

Here $$\vec{V}_p = \left(\frac{\vec{V}_1 \cdot \vec{V}_2}{\|\vec{v}_1\|^2}\right)\vec{V}_1 = \frac{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}4\\3\\-1\end{bmatrix}}{(\sqrt{3})^2}\vec{V}_1 = \frac{6}{3}\vec{V}_1 = 2\vec{V}_1$$

$\vec{V}_p = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. And since $\vec{V}_p$ is based at $(-1,-2,-1)$, the head of $\vec{V}_p$ is at

$$(-1+2,-2+2,-1+2) = (1,0,1)$$

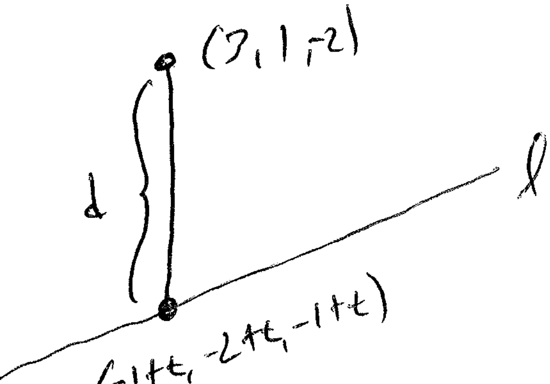
as before.

The rest of the solution follows like Method 1. ∎

## Method 3

**Strategy**: Minimalize the distance between $(3,1,-2)$ and the line $\ell$ via the Euclidean distance function. Once set up this function will only be a function of $t$, so this is an optimization problem from calculus 1.

**Solution**




Define $f : \mathbb{R} \to \mathbb{R}$, where

$$f(t) = \sqrt{(3-(-1+t))^2 + (1-(-2+t))^2 + (-2-(-1+t))^2}$$
$$= \sqrt{(4-t)^2 + (3-t)^2 + (-1-t)^2}$$

This is a Calc I problem. You can finish it, as you now know the answer.