A CHAIN RULE EXAMPLE.

110.202 CALCULUS III
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Here is a problem that I made up on the fly in my office hour:

Exercise. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y,z) = xyz^2$. If we knew that $x = s+t$, $y = s^2 - t$, and $z = st$ were all functions of the variables s and t, how is f changing with respect to the variable s.

Note. *There are at least two ways to solve this problem. One method is to directly substitute the functions for x, y and z, in terms of s and t into f, and then simply calculate the partial of f with respect to s. The other way is use teh Chain Rule. But in either case, we are composing the three coordinate functions, as a set, with f to create a new function. SO to set up this problem, we first set up the composition correctly:*

Create a new function $\mathbf{g}\colon \mathbb{R}^2 \to \mathbb{R}^3$, *where* $\mathbf{g}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} s+t \\ s^2-t \\ st \end{bmatrix}$. *Then the composition is the function* $h\colon \mathbb{R}^2 \to \mathbb{R}$, *where*

$$h(s,t) = f\left(\mathbf{g}(s,t)\right) = f\left(x(s,t).y(s,t), z(s,t)\right) = f(s+t, s^2-t, st),$$

and the problem is asking for the quantity $\dfrac{\partial h}{\partial s}(s,t)$, *namely, how the composition of f with the three coordinate functions is changing with resepct to the coordinate s.*

Method 1: direct substitution.

Strategy. Substitute directly the functions $x = s+t$, $y = s^2 - t$, and $z = st$ into f, so that f is now a function of s and t, and then calculate the partial derivative with respect to s.

Solution. Again, call $h(s,t) = f\left(\mathbf{g}(s,t)\right)$. Then

$$\begin{aligned} h(s,t) &= f\left(x(s,t).y(s,t), z(s,t)\right) = f(s+t, s^2-t, st) \\ &= (s+t)(s^2-t)(st)^2 = s^3 - st + s^2t - t^2)s^2t^2 \\ &= s^5t^2 - s^3t^3 + s^4t^3 - s^2t^4. \end{aligned}$$

Then

$$\frac{\partial h}{\partial s}(s,t) = 5s^4t^2 - 3s^2t^3 + 4s^3t^3 - 2st^4.$$

Method 2: Via the Chain Rule.

Strategy. We employ the Chain Rule directly to calculate $Dh(s,t)$ as the product of the two derivatives $Df(\mathbf{g}(s,t))$ and $D\mathbf{g}(s,t)$. Then simply take the first element $\frac{\partial h}{\partial s}(s,t)$ of $Dh(s,t)$ adn write it out in terms of the matrix calculation.

Solution. Using the Chain Rule, we have

$$\begin{aligned} Dh(s,t) &= Df\left(\mathbf{g}(s,t)\right)D\mathbf{g}(s,t) \\ &= \begin{bmatrix} \frac{\partial f}{\partial x}\left(\mathbf{g}(s,t)\right) & \frac{\partial f}{\partial y}\left(\mathbf{g}(s,t)\right) & \frac{\partial f}{\partial z}\left(\mathbf{g}(s,t)\right) \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial s}(s,t) & \frac{\partial x}{\partial t}(s,t) \\ \frac{\partial y}{\partial s}(s,t) & \frac{\partial y}{\partial t}(s,t) \\ \frac{\partial z}{\partial s}(s,t) & \frac{\partial z}{\partial t}(s,t) \end{bmatrix}. \end{aligned}$$

Then $\frac{\partial h}{\partial s}(s,t)$ is just the first row of the first matrix multiplied by the first column of the second:

$$\begin{aligned} \frac{\partial h}{\partial s}(s,t) &= \frac{\partial f}{\partial x}\left(\mathbf{g}(s,t)\right)\cdot\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\left(\mathbf{g}(s,t)\right)\cdot\frac{\partial y}{\partial s}(s,t) + \frac{\partial f}{\partial z}\left(\mathbf{g}(s,t)\right)\cdot\frac{\partial z}{\partial s}(s,t) \\ &= \left(yz^2\Big|_{\substack{x=s+t\\ y=s^2-t\\ z=st}} \right)\cdot 1 + \left(xz^2\Big|_{\substack{x=s+t\\ y=s^2-t\\ z=st}} \right)\cdot 2s + \left(2xyz\Big|_{\substack{x=s+t\\ y=s^2-t\\ z=st}} \right)\cdot t \\ &= (s^2-t)(st)^2 + (s+t)(st)^2(2s) + 2(s+t)(s^2-t)(st)t \\ &= s^4t^2 - s^2t^3 + 2s^4t^2 + 2s^3t^3 + 2s^4t^2 - 2s^2t^3 + 2s^3t^3 \\ &= 5s^4t^2 - 3s^2t^3 + 4s^3t^3 - 2st^4. \end{aligned}$$