Question 1. [15 points] Define $h(x,y) = \begin{cases} \dfrac{\ln(2-x)-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

Do the following:

(a) Determine the domain for h.

Strategy: Look for and record any places where the numerator and/or denominator are not defined or the denominator equals 0.

Solution: First, note that the denominator is defined on all of $\mathbb{R}^2$ and equals 0 only at at the origin. But the function is defined separately at $(0,0)$ as 0. Hence the denominator is defined at all points of $\mathbb{R}^2$. The numerator is defined for all y and for $x < 2$, since the natural logarithm is defined only when its argument is strictly greater than 0. Hence

$$\text{the domain of } h(x,y) = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x < 2\right\}.$$

Grading notes: First, it must be explicit that the domain of h involves points of the plane. Stating the domain in terms of only one variable is a serious misconception. Secondly, care must be taken somehow to state that all values for the variable y are acceptable. Third, stating that the function is not defined at the origin is also a mistake, as the function is separately defined there. And lastly, this part was worth 3 points, total.

(b) Show that h is not continuous at $(0,0)$.

Strategy: We check certain directions of approach to see if the limit along these directions either doesn't exist or if the limits from different directions are not equal. However, if any of the directional limits doesn't exist or exists but is not equal to the function's value at the origin, then the function cannot be continuous at the origin. If the functions is continuous at the origin, then every and all directional limits there will necessarily exist and equal the function value.

Solution: First, we approach the origin along the x-axis, where $y = 0$. Then

$$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{x\to 0} \frac{\ln(2-x)-0^2}{x^2+0^2} = \infty,$$

(Stating this is true will generate some points. But evaluating this limit to be ∞ must include a justification to generate full credit. Some reasoning is necessary since it is not immediately clear that the two side limits are both ∞ or not, and both the numerator and the denominator are functions of x. One justification would be that the numerator is bounded, with a limit of $\ln 2 > 0$, while the denominator has a limit of 0. Hence the fraction is unbounded as one approaches $(0,0)$. Also, L'Hopital's Rule does not apply here since there is no indeterminate form involved. At this point, we can stop, since the limit at $(0,0)$ from one direction doesn't equal the function value. An explicit statement to this effect, concluding that h is not continuous at $(0,0)$, completes the solution.
If one started by checking the limit along the y-axis, where $x = 0$, we would get

$$\lim_{(x,y)\to(0,0)} h(x,y) = \lim_{y\to 0} \frac{(\ln(2)-y^2}{0^2+y^2} = \infty.$$

Again, stating this will generate some credit, but a reason is necessary to gain full credit. Also here, L'Hopital's Rule does not apply. **Grading notes:** This problem was worth 6 points total. Also, attempting to evaluate the full 2-dimensional limit by direct substitution will not work here since (1) the very definition of a limit does not take into account the function's value there (unless you already know that the function is continuous, and (2) an attempt at direct substitution can be shown to confer an incorrect value for the limit.

(c) Calculate the derivative of h evaluated at the point $(x,y) = (1,1)$, if it exists, or explain why it does not exist there.

Strategy: Calculate the two partial derivatives of h, evaluate at $(1,1)$ and construct the 1×2-matrix with these components.

Solution: Here,

$$\begin{aligned}
\frac{\partial h}{\partial x}(x,y) &= \frac{\frac{\partial}{\partial x}\left[\ln(2-x) - y^2\right](x^2+y^2) - (\ln(2-x)-y^2)\frac{\partial}{\partial x}\left[x^2+y^2\right]}{(x^2+y^2)^2} \\
&= \frac{\frac{-1}{2-x}(x^2+y^2) - (\ln(2-x)-y^2)(2x)}{(x^2+y^2)^2} \\
\frac{\partial h}{\partial y}(x,y) &= \frac{\frac{\partial}{\partial y}\left[\ln(2-x) - y^2\right](x^2+y^2) - (\ln(2-x)-y^2)\frac{\partial}{\partial y}\left[x^2+y^2\right]}{(x^2+y^2)^2} \\
&= \frac{(-2y)(x^2+y^2) - (\ln(2-x)-y^2)(2y)}{(x^2+y^2)^2}.
\end{aligned}$$

And

$$Dh(1,1) = \begin{bmatrix} \frac{\partial h}{\partial x}(1,1) & \frac{\partial h}{\partial y}(1,1) \end{bmatrix} = \begin{bmatrix} \frac{(-1)(2)-(0-1)(2)}{4} & \frac{(-2)(2)-(0-1)(2)}{4} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \end{bmatrix}.$$

Grading notes: This problem was worth 6 points total. Basically, calculating each of the two partial derivatives generates 4 points, and correctly constructing the derivative is the other two. Variations on this generated some variations on the grading. For example, stating that the two partials were not equal as a claim that the derivative does not exist is not valid. Nor was valculating the partial derivatives without regard to the fact that the function is a ratio of functions.