$\boxed{\text{Third}}$) It helps solve ODEs

example Solve the IVP $y''-3y'-4y=0$, $y(0)=1$

$y'(0)=2$.

Solution: Since LT is linear, we can transform the entire equation:

$$\mathcal{L}\{y''-3y'-4y=0\}$$

$$\mathcal{L}\{y''-3y'-4y\}=\mathcal{L}\{0\}.$$

$$\underbrace{\mathcal{L}\{y''\}}-3\underbrace{\mathcal{L}\{y'\}}-4\mathcal{L}\{y\}=0$$

$$s^2\mathcal{L}\{y\}-sy(0)-y'(0)-3\left[s\mathcal{L}\{y\}-y(0)\right]-4\mathcal{L}\{y\}=0$$

$$s^2\mathcal{L}\{y\}-s-2-3s\mathcal{L}\{y\}+3-4\mathcal{L}\{y\}=0$$

$$\Longrightarrow \quad \mathcal{L}\{y\}=\frac{s-1}{s^2-3s-4}=\frac{s-1}{(s-4)(s+1)}=\frac{a}{s-4}+\frac{b}{s+1}$$

solve by partial fraction decomp to get

$$\mathcal{L}\{y\}=\frac{3}{5}\left(\frac{1}{s-4}\right)-\frac{2}{5}\left(\frac{1}{s+1}\right)$$

What functions have these as LT?

$$\mathcal{L}\{e^{4t}\}=\frac{1}{s-4}, \quad \mathcal{L}\{e^{-t}\}=\frac{1}{s+1}.$$

$$\Longrightarrow \quad y(t)=\frac{3}{5}e^{4t}+\frac{2}{5}e^{-t}$$

Now solve it the other way in your notes!

Advantages?

① Works equally well with higher order ODEs.

② Turns an ODE into an algebraic equation

③ Many basic functions have well-known LTs (chart page 317)

④ Incorporates both nonhomogeneous forces and initial data into the calculation,

ex. Solve $\ddot{x}+3\dot{x}+2x=12e^{2t}$ $x(0)=1$, $\dot{x}(0)=-1$.

Here $\mathcal{L}\{\ddot{x}+3\dot{x}+2x=12e^{2t}\}$.

$$\mathcal{L}\{\ddot{x}\}+3\mathcal{L}\{\dot{x}\}+2\mathcal{L}\{x\}=12\left(\frac{1}{s-2}\right)$$

Finish this calculation:

The solution is $x(t)=e^{2t}+3e^{-2t}-3e^{-t}$

ex. Solve $y^{(4)}-y=0$, $y(0)=y''(0)=y'''(0)=0$, $y'(0)=1$.

Here $\mathcal{L}\{y^{(4)}-y=0\}$ becomes

$$s^4\mathcal{L}\{y\}-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)-\mathcal{L}\{y\}=0$$

$$\Rightarrow \mathcal{L}\{y\}=\frac{s^2}{s^4-1}=\frac{s^2}{(s^2-1)(s^2+1)}=\frac{as+b}{s^2-1}+\frac{cs+d}{s^2+1}$$

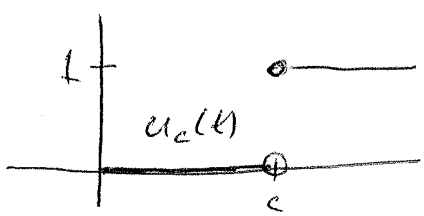
Note: $a=c=0$, $b=d=\frac{1}{2}$

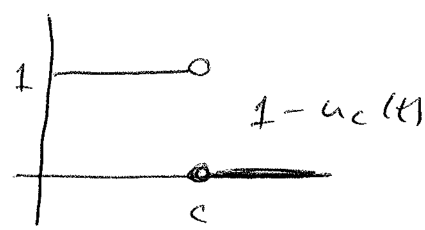
$$=\frac{1}{2}\left(\frac{1}{s^2-1}\right)+\frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

Hence $\boxed{y(t)=\frac{1}{2}\sinh t+\frac{1}{2}\sin t}$

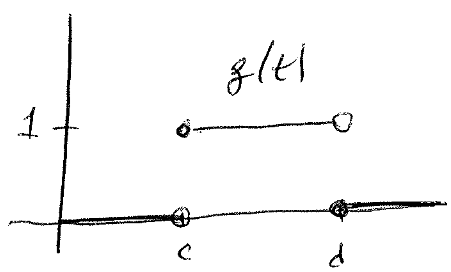
⑤ Forcing functions (the ~~nonhomo~~ $g(t)$ that makes a linear ODE nonhomogeneous), are not always continuous. LTs are good for this.

Define $u_c(t) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$ a type of Heaviside function. $(c=0)$ Then $1 - u_c(t)$ is

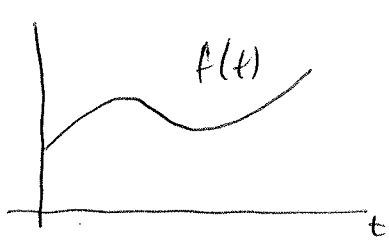


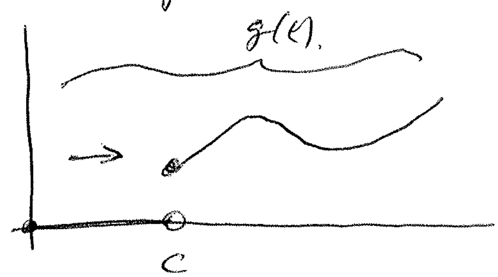


Then a pulse on (c,d) would be $g(t) = u_c(t) - u_d(t)$



And any modulated signal $f(t)$ ~~delayed~~ "turned on" @ $t = c > 0$ could be written $g(t) = u_c(t) f(t-c)$





Some results for step functions and pulses

(I) $\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$

$= \lim_{A\to\infty} \int_c^A e^{-st}\,dt = \lim_{A\to\infty} \frac{e^{-sA}}{-s} - \left(\frac{e^{-cs}}{-s}\right)$

$= \frac{e^{-cs}}{s}, \quad s>0$

(II) $\mathcal{L}\{u_c(t) - u_d(t)\} = \frac{e^{-cs}}{s} - \frac{e^{-ds}}{s} = \frac{e^{-cs} - e^{-ds}}{s}, \quad s>0.$

(III) $\mathcal{L}\{g(t)\} = \mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$

where $\mathcal{L}\{f(t)\} = F(s)$, for $s > a \geq 0$.

Why?

$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt$

$= \int_c^\infty e^{-st} f(t-c)\,dt$

$= \int_0^\infty e^{-s(t+c)} f(t)\,dt$

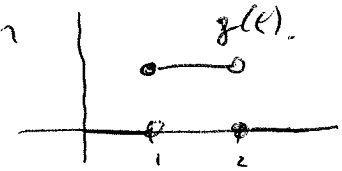
$= e^{-sc} \int_0^\infty e^{-st} f(t)\,dt = e^{-sc} F(s)$

when $s > a$ as long as $a \geq 0$.

Here a comes from $F(s)$.

example Solve the IVP $y'' - 3y' - 4y = g(t)$

where $g(t)$ is the unit pulse function



$$g(t) = u_1(t) - u_2(t),$$

and $y(0) = 1$, and $y'(0) = 2$.

Solution Note: One could think of this problem as 3 separate ODEs (IVPs actually): as follows:

ⓐ Solve the IVP $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$ and then evaluate the endpoint of the solution @ $t = 1$. ~~Then~~ Call these $y(1) = y_1$, $y'(1) = y_1'$.

ⓑ Solve the new IVP $y'' - 3y' - 4y = 1$, $y(1) = y_1$, $y'(1) = y_1'$. Then evaluate the solution to this ODE @ $t = 2$. call it $y(2) = y_2$, $y'(2) = y_2'$

ⓒ Solve again the ODE $y'' - 3y' - 4y = 0$ with new initial conditions $y(2) = y_2$, $y'(2) = y_2'$.

ⓓ Stitch together your solutions to a single function.

Laplace Transform solution

Transform the equation to $\mathcal{L}\{y''-3y'-4y\} = u_1(t)-u_2(t)\}$

$$\mathcal{L}\{y''-3y'-4y\} = \mathcal{L}\{u_1(t)-u_2(t)\}.$$

$$(s^2-3s-4)\,\mathcal{L}\{y\}+1-s = \frac{e^{-s}}{s}-\frac{e^{-2s}}{s} = \frac{e^{-s}-e^{-2s}}{s}.$$

as in the previous example on the left hand side.

Solve for $\mathcal{L}\{y\}$ to get

$$\mathcal{L}\{y\} = \left(\frac{e^{-s}-e^{-2s}}{s}+s-1\right)\left(\frac{1}{s^2-3s-4}\right)$$

$$= \frac{e^{-s}-e^{-2s}}{s(s-4)(s+1)} + \frac{s-1}{(s-4)(s+1)}$$

Note that the latter term is just the Laplace transform of the homogeneous solution, and decomposes to

$$\frac{s-1}{(s-4)(s+1)} = \frac{3}{5}\left(\frac{1}{s-4}\right)+\frac{2}{5}\left(\frac{1}{s+1}\right)$$

Using a partial fraction decomposition on the other term, we get

$$(e^{-s}-e^{-2s})\left(\frac{1}{s(s-4)(s+1)}\right) = (e^{-s}-e^{-2s})\left(-\frac{1}{4}\left(\frac{1}{s}\right)+\frac{1}{20}\left(\frac{1}{s-4}\right)+\frac{1}{5}\left(\frac{1}{s+1}\right)\right)$$

Put this together to get

$$\mathcal{L}\{y\} = e^{-s}\left(-\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{20}\left(\frac{1}{s-4}\right) + \frac{1}{5}\left(\frac{1}{s+1}\right)\right)$$
$$- e^{-2s}\left(-\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{20}\left(\frac{1}{s-4}\right) + \frac{1}{5}\left(\frac{1}{s+1}\right)\right)$$
$$+ \frac{3}{5}\left(\frac{1}{s-4}\right) + \frac{2}{5}\left(\frac{1}{s+1}\right)$$

Recall that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, and $\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}$,

and $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}F(s)$.

Hence on each of the summands above, we get

$$y(t) = u_1(t)\left(-\frac{1}{4} + \frac{1}{20}e^{4(t-1)} + \frac{1}{5}e^{-(t-1)}\right)$$
$$- u_2(t)\left(-\frac{1}{4} + \frac{1}{20}e^{4(t-2)} + \frac{1}{5}e^{-(t-2)}\right)$$
$$+ \frac{3}{5}e^{4t} + \frac{2}{5}e^{-t}.$$

Messy, sure! But graph this function (you will need a log plot to really see it). It is a smooth function, even at the sts $t=1$ and $t=2$, and solves $y'' - 3y' - 4y = u_1(t) - u_2(t)$, $y(0) = 1$, $y'(0) = 2$.