~~Welcome~~ 110.302 Lecture 34 : 11/20/15

Recall the improper Integral

$$\int_a^\infty f(t)\,dt = \lim_{A\to\infty}\int_a^A f(t)\,dt$$

Here LHS converges if limit exists, diverges if not

ex. $\int_a^\infty 4\,dt$ div $\forall a\in\mathbb{R}$

ex $\int_0^\infty e^{-kt}\,dt = \lim_{A\to\infty}\int_0^A e^{-kt}\,dt = \lim_{A\to\infty}\left[\frac{e^{-kA}}{-k} + \frac{1}{k}\right]$

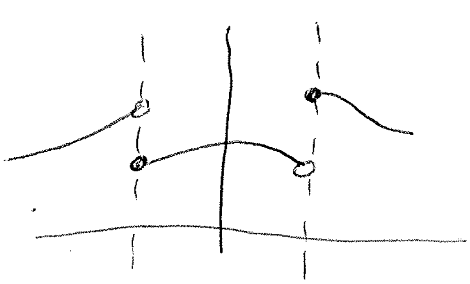
$= \frac{1}{k}$ converges when $k>0$.

Recall $f(t)$ is piecewise continuous on $\alpha\le t\le\beta$

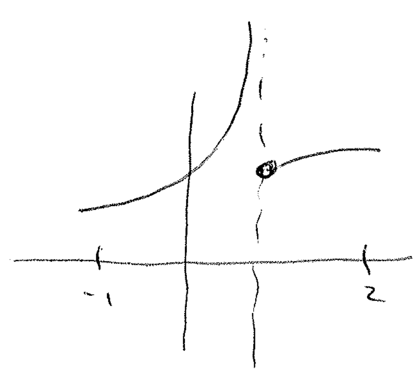
if $\exists$ partition $\alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = \beta$

where ① $f(t)$ is continuous on each (t_{i-1}, t_i)

② Both one-sided limits exist on each (t_{i-1}, t_i).



piecewise cont



not piecewise continuous on $(-1,2)$.

Def A transform (integral) is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t)\, dt$$

Here $K(s,t)$ is called the kernel of the transform and $\alpha \geq -\infty$, $\beta \leq \infty$. $F(s)$ becomes the transformed $f(t)$.

Reason to transform: Certain problems become easier to solve....

ex. (Not a transform, but...)

Calculate $y'(x)$, for $y(x) = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$, $x>0$.

Idea: Since $y>0$, for $x>0$, instead 1st change problem:

Logarithmic Differentiation: Given

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2),$$

$$\frac{d}{dx}[\ln y] = \frac{y'}{y} = \underbrace{\frac{3}{4x} + \frac{2x}{2(x^2+1)} - \frac{15}{3x+2}}$$

$$\Longrightarrow \quad y' = y(\quad) = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Transforms have a similar effect

Def Laplace Transform

Let $f(t)$ be defined for $t \geq 0$ with some "added properties".

$$\Longrightarrow \quad \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

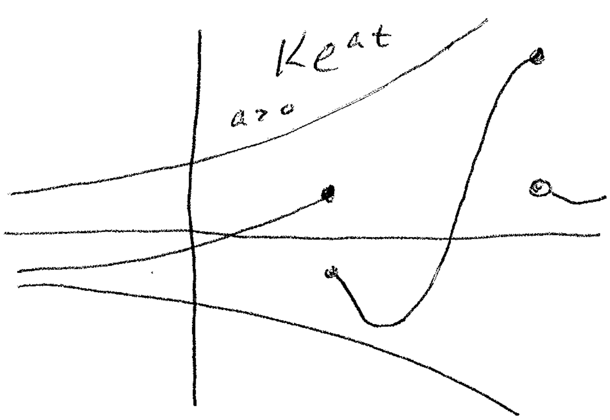
is the Laplace Transform of $f(t)$ whenever the integral converges.

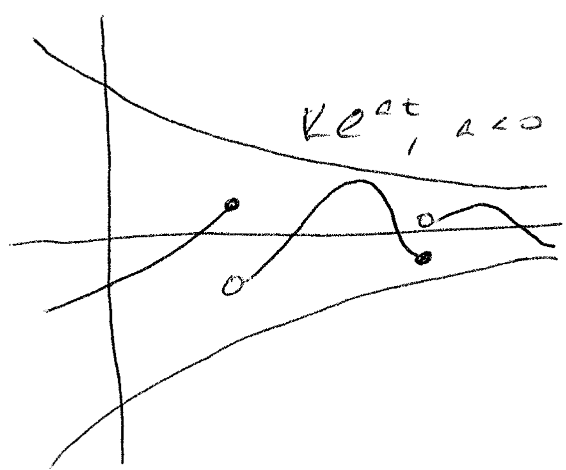
Notes ① In general, s is a complex variable, but for now, keep it real.

② "added properties":

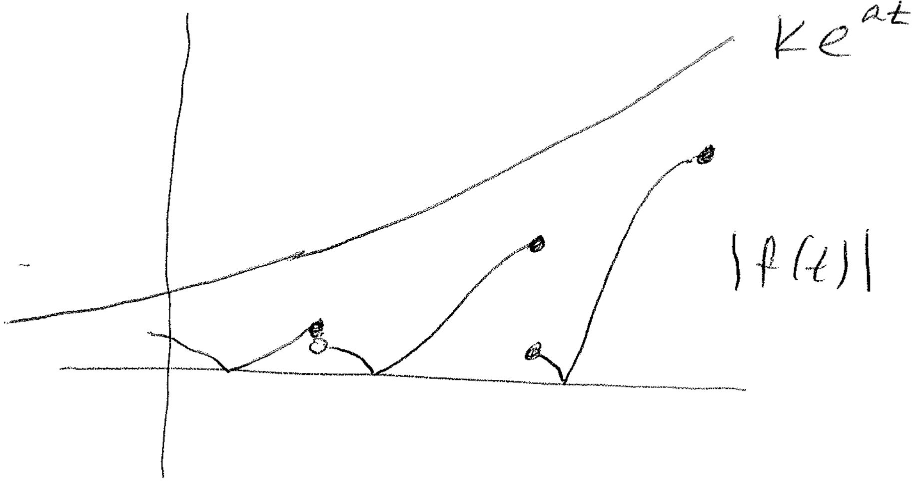
ⓐ f is piecewise cont. on $[0, A]$ for $A > 0$.

ⓑ $|f(t)| \leq Ke^{at}$, for $t \geq M$, where $K > 0$, $M > 0$, a are all real.





For ⓑ, we only require that f does not increase faster than exponentially.



Thm If $f(t)$ is defined so that ⓐ & ⓑ hold, then

$$\mathcal{L}\{f(t)\} = F(s)$$

exists for $s > a$.

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pf. Need to show $\int_0^\infty e^{-st} f(t)\,dt$ converges.

Here
$$\int_0^\infty e^{-st} f(t)\,dt = \underbrace{\int_0^M e^{-st} f(t)\,dt}_{\text{converges by ⓐ}} + \int_M^\infty e^{-st} f(t)\,dt$$

$$\ldots \le \int_M^\infty e^{-st} f(t)\,dt \le \left| \int_M^\infty e^{-st} f(t)\,dt \right| \le \int_M^\infty |e^{-st} f(t)|\,dt = \int_M^\infty e^{-st} |f(t)|\,dt$$

$$\le \int_M^\infty e^{-st} K e^{at}\,dt = K \int_M^\infty e^{(a-s)t}\,dt$$

which conv as long as $s > a$. ∎

examples of Laplace Transforms

(I) Let $f(t) \equiv 1$, $t > 0$:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty}\int_0^A e^{-st}\,dt$$

$$= \lim_{A\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^A = \lim_{A\to\infty} \frac{e^{-sA}-1}{-s} = \frac{1}{s}, \quad s>0$$

So for $f(t) \equiv 1$, $t > 0$, $F(s) = \frac{1}{s}$, $s > 0$.

Q: How does this change for $f(t) = k$, $t > 0$, $k \in \mathbb{R}$.

(II) Let $f(t) = e^{at}$, $t \geq 0$:

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}, \quad s>a$$

Notes (1) Domains are important: (II) only works for $s > a$!

(2) Notice how (II) is (I) when a is chosen to be 0.

(III) Harder stuff

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st}\sin at\,dt$$

$$= \lim_{A\to\infty}\int_0^A e^{-st}\sin at\,dt$$

By Integration by parts

$$F(s) = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt$$

Another Integration by Parts

$$\mathcal{L}\{\sin at\} = \underbrace{F(s) = \frac{1}{a} - \frac{s^2}{a^2} F(s)}$$

or

$$F(s) = \frac{a}{s^2+a^2}, \quad s > 0$$

Q: Why is LT useful?

$\boxed{\text{First,}}$ $\underline{\mathcal{L}\{f(t)\} \text{ is an operator}}$, and it is linear (integral operators are linear).

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\,dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt$$

$$= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.$$

$\boxed{\text{Second}}$ It <u>behaves well with derivatives</u>

<u>Thm</u> Suppose $f(t)$ is cont on $t \in [0, \infty)$ and $f'(t)$ piecewise cont. there, and $|f(t)| \le K e^{at}$, $t \ge M$

$$\Rightarrow \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

on $s > a$.

Quick proof

pf. Given $f(t)$, $t>0$, continuous,

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st} f'(t)\,dt$$

$$\overset{\text{Int parts}}{=} \lim_{A\to\infty}\left(e^{-st} f(t)\Big|_0^A - \int_0^A (-s) e^{-st} f(t)\,dt \right)$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\mathcal{L}\{f(t)\} - f(0) \qquad \blacksquare$$

Note: Can also recursively apply this formula to get

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

up to

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

provided all der $f(t), \dots, f^{(n-1)}(t)$ are cont on $t \geq 0$, $f^{(n)}(t)$ is piecewise cont. and all derivatives satisfy

$$|f^{(i)}(t)| \leq Ke^{at}, \quad t \geq M$$

for $i = 0, \dots, n-1$.