110.302 Lecture 31. 11/13/15

There is a much better way to characterize this:

<u>Proposition</u> A system $\vec{x}' = \vec{f}(\vec{x}) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$ is almost linear at an isolated critical pt $\vec{x} = \vec{x}^0$ if both F, G have continuous partials up to order 2, and $Df(\vec{x}) = \begin{bmatrix} \frac{\partial F}{\partial x}\Big|_{\vec{x}^0} & \frac{\partial F}{\partial y}\Big|_{\vec{x}^0} \\ \frac{\partial G}{\partial x}\Big|_{\vec{x}^0} & \frac{\partial G}{\partial y}\Big|_{\vec{x}^0} \end{bmatrix}$

is non-degenerate.

Notes ① Here, $Df(\vec{x})$ is the derivative of f as a vector-valued function on a domain in $\mathbb{R}^2$, called the Jacobian of f at $\vec{x}^0$. It becomes the matrix of the associated linearized system.

② Indeed, change variables to "move" $\vec{x}^0$ to the origin: let $\vec{u} = \vec{x} - \vec{x}^0 = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$, $\vec{u}' = \vec{x}'$

So let $\vec{x}' = \vec{f}(\vec{x})$ have a critical pt at $\vec{x}^0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

Then for $u_1 = x - x_0$, $u_2 = y - y_0$, we know

$u_1' = x'$, $u_2' = y'$, and.

Then

$$\vec{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} F(x-x_0, y-y_0) \\ G(x-x_0, y-y_0) \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{\cancel{F(x_0,y_0)}^{0} + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)} + \overbrace{\eta_1(x,y)} \\ \underbrace{\cancel{G(x_0,y_0)}^{0} + G_x(x_0,y_0)(x-x_0) + G_y(x_0,y_0)(y-y_0)}_{1^{st}\text{ Taylor Polynomial of } f \text{ at } \vec{x}^0} + \underbrace{\eta_2(x,y)}_{\text{Higher order stuff.}} \end{bmatrix}$$

$$= \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \begin{bmatrix} \eta_1(x,y) \\ \eta_2(x,y) \end{bmatrix}$$

$$\vec{u}' = A\vec{u} + \overrightarrow{g(\vec{x})}$$

and $\overrightarrow{g(\vec{x})}$ will satisfy the smallness property since 2^{nd} derivatives also exist.

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ex. Back to example $\begin{matrix} \dot{x} = 4-2y \\ \dot{y} = 12-3x^2 \end{matrix}$. Here there is a fixed pt at $(2,2)$. Transform $\begin{matrix} u = x-2 \\ v = y-2. \end{matrix}$

Here $\dot{u} = \dot{x}$, $\dot{v} = \dot{y}$, and

$$\dot{u} = -4-2(v+2) = -2v$$
$$\dot{v} = 12-3(u+2)^2 = -12u-3u^2$$

Then $\vec{u}' = A\vec{u} + \overrightarrow{g(\vec{u})} = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}\vec{u} + \begin{bmatrix} 0 \\ -3u^2 \end{bmatrix}$

So $A = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$, $\overrightarrow{g(\vec{u})} = \begin{bmatrix} 0 \\ -3u^2 \end{bmatrix}$.

Since $\|\overrightarrow{g(\vec{u})}\| = \sqrt{0^2 + (-3u^2)^2} = 3u^2$, and

$\|\vec{u}\| = \sqrt{u^2+v^2} = r$, where $\begin{matrix} u = r\cos\theta \\ v = r\sin\theta \end{matrix}$, we have

$$\lim_{\vec{u}\to\vec{0}} \frac{\|\vec{g}(\vec{u})\|}{\|\vec{u}\|} = \lim_{r\to 0} \frac{3r^2\cos^2\theta}{r} = \lim_{r\to 0} 3r\cos\theta = 0$$

Hence system is almost linear at $(2,2)$.

<u>Thm</u> Let r_1, r_2 be the eigenvalues of the linear system $\vec{x}' = A\vec{x}$ corresponding to the almost linear $\vec{x}' = A\vec{x} + \overrightarrow{g(\vec{x})}$. Then

1. If $r_1 \neq r_2$ not purely imaginary, the stability and type of the fixed pts are the same.
2. If $r_1 = r_2$, the fixed pt of the nonlinear system has the same stability as that of the linear system but not necessarily the type.
3. If $r = \pm i\mu$, then neither the stability nor the type of the nonlinear critical pt can be determined by the linear system.

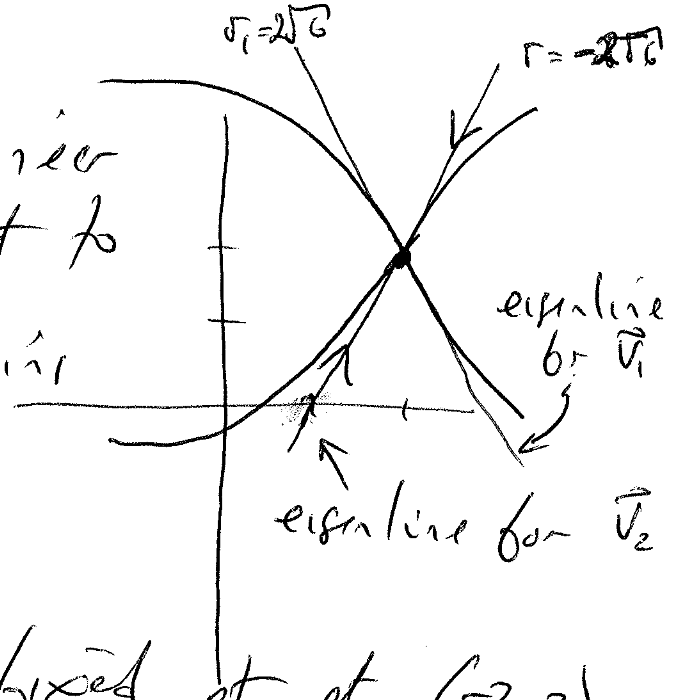
In the example, $A = \begin{bmatrix} 0 & -2 \\ -12 & 0 \end{bmatrix}$ has eigenvalues

$r = \pm 2\sqrt{6}$.

Hence the linear system has a saddle at the origin. By the Thm, then, so does the original nonlinear system at $(2,2)$.

Furthermore, for $r_1 = 2\sqrt{6}$, $r_2 = -2\sqrt{6}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -\sqrt{6} \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ \sqrt{6} \end{bmatrix}$

To linear approximation, even the phase portraits are the same. The eigenlines of the linear system are tangent to the level set passing through $(2,2)$.



exercise: Do this for the fixed pt at $(-2,2)$. The linear system has a center at the origin. But by the Thm we cannot tell if the original is a center or not.

Note: As long as F, G are "nice" (have cont. derivatives up to and including order 2) at a critical pt, the system is almost linear. One calculates $A = \begin{bmatrix} F_x|_{\vec{x}^0} & F_y|_{\vec{x}^0} \\ G_x|_{\vec{x}^0} & G_y|_{\vec{x}^0} \end{bmatrix}$ and there is no need to worry about the $\vec{g}(\vec{x})$.

ex. Determine the stability of $\vec{x}^0 = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$ of $\begin{aligned} \dot{x} &= (1+x)\sin y \\ \dot{y} &= 1 - x - \cos y \end{aligned}$

<u>Soln</u>: Both $\begin{aligned} F(x,y) &= (1+x)\sin y \\ G(x,y) &= 1 - x - \cos y \end{aligned}$ are C^∞, so system is almost linear everywhere.

Here ~~[illegible]~~ $F(2,\pi) = 0 = G(2,\pi)$, and

$$A = \begin{bmatrix} F_x(2,\pi) & F_y(2,\pi) \\ G_x(2,\pi) & G_y(2,\pi) \end{bmatrix} = \begin{bmatrix} \sin\pi & (1+2)\cos\pi \\ -1 & \sin\pi \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues of A satisfy $r^2 - 0r + 3 = 0$ or $r = \pm\sqrt{3}$

Hence $\vec{x}^0 = \begin{bmatrix} 2 \\ \pi \end{bmatrix}$ is a saddle and unstable.