110.302 Lecture 30: 11/11/15

XII

Lastly, a different viewpoint in phase plane analysis:

ex. Solve the system $\dot{x} = 4 - 2y$

$$\dot{y} = 12 - 3x^2$$

Note: This system is nonlinear and nonhomogeneous (why?). It is still autonomous.

<u>Solution</u>: We can easily calculate the critical pts here. $4-2y=0$ only when $y=2$. And $12-3x^2=0$ when $x=\pm 2$. Hence $x=2, y=2$, and $x=-2, y=2$ are the only critical pts.

But this is limited information.

Another tack? Remove the parameter t from the solutions by writing y directly (implicitly) in terms of x: This involves a creative use of the Chain Rule (in Leibniz notation)

Suppose we wanted to parameterize a curve given by $y = y(x)$. Write x as some function of t. Then y is also a function of t given by $y(t) = y(x(t))$. If diff, then $y'(t) = y'(x(t)) \cdot x'(t)$, or in Leibniz notation $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$, so $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

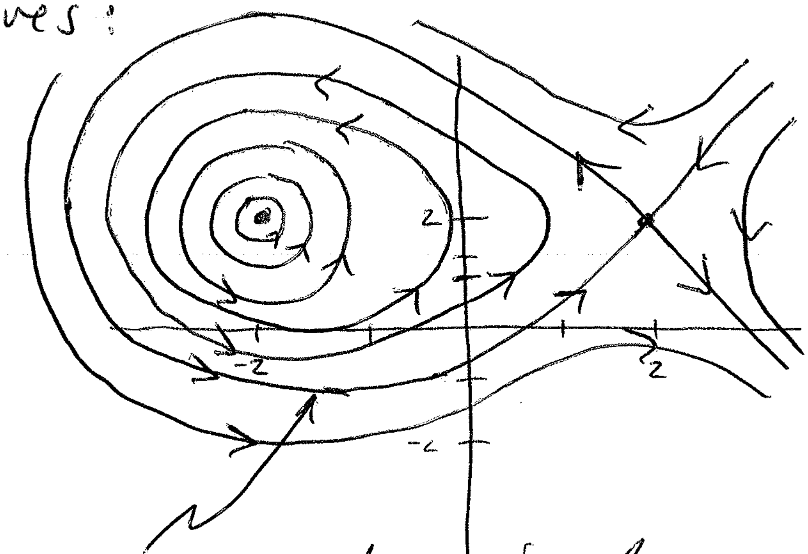
We can reverse this process and write this system of ODE into a single 1^{st} order ODE: $\dot{x} = \frac{dx}{dt} = 4 - 2y$ $\quad \dot{y} = \frac{dy}{dt} = 12 - 3x^2$ into

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12 - 3x^2}{4 - 2y}$$

One may solve this as an exact ODE given by $\frac{dy}{dx} - \frac{12 - 3x^2}{4 - 2y} = 0$, or $(4 - 2y)\frac{dy}{dx} - (12 - 3x^2) = 0$

solved to give $\varphi(x, y) = 4y - y^2 - 12x + x^3 = c$.

Level sets of $\psi(x,y)$ correspond to solution curves:

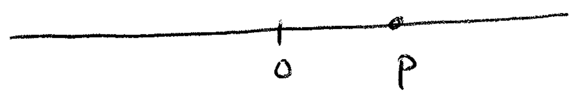


more general version of a separatrix

How to analyze the behavior of solutions near a critical pt of a nonlinear system:

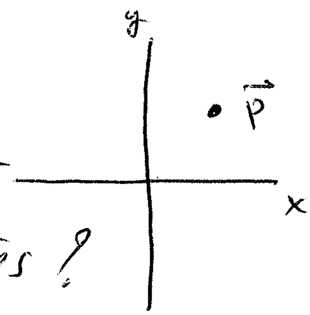
<u>Step 1</u>: Understand the persistence of phase portraits under perturbations (small changes).

- Choose a pt $p \in \mathbb{R}$.



Vary it slightly and randomly.
Depending on your choice of p, does a slight change possibly alter its parity?
Yes, but only if $p = 0$.

- Choose a pt $\vec{p} \in \mathbb{R}^2$: Can a small random change alter the parity of its coordinates?



Yes, but only if $\vec{p}$ is on one of the axes!

This is the notion of <u>stability</u> under small perturbations:
$\vec{p}$ is considered stable under perturbations iff neither coord. of $\vec{p}$ is 0.

Step 1: cont'd.

Now let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with eigenvalues

$$r = \frac{-(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

where $p = a+d = \operatorname{tr} A$, $q = ad-bc = \det A$.

If we perturb the entries of A slightly, we perturb also p and q slightly.

What is the effect?

Sinks, saddles, sources

Type and stability persist

ⓐ If $r<0$ or $r>0$, perturbed will stay the same.

ⓑ If $r_1 \neq r_2$ This will also persist.

ⓒ If $r = \lambda \pm i\mu$ and $\lambda > 0$ or $\lambda < 0$, will persist.

Stars and improper nodes

Type will not persist but stability will

ⓓ If $r_1 = r_2$ This will not persist under random perturbations.

But the fact that $r_1 > 0$ or $r_1 < 0$ will.

Centers and such

Neither type nor stability will persist.

ⓔ If $r = \lambda \pm i\mu$ and $\lambda = 0$, probably will not persist. May go $\lambda > 0$ or $\lambda < 0$.

ⓕ If $r = 0$. May go positive or negative.

Q/ So how do solutions to $\vec{x}' = \vec{f}(\vec{x}) = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$ near a critical pt $\vec{p} \in \mathbb{R}^2$ behave?

A/ Very much like that of a linear system when the linear system is stable under perturbations:

~~Notes ① With a change in variables, $\vec{u} = \vec{x} - x$~~

Let $\vec{x}' = \vec{f}(\vec{x})$ have a critical pt $\vec{p} \in \mathbb{R}^2$.

Notes ① With a change in variables $\vec{u} = \vec{x} - \vec{p}$, the new system will ~~look~~ remain nonlinear but with a critical pt @ $\vec{u} = \vec{0}$ in $\vec{u}$-space.

② Very close to $\vec{u} = \vec{0}$, $\vec{u} = \vec{f}(\vec{u})$ looks like a perturbed $\vec{u} = A\vec{u}$ for some constant matrix A.

When ② is so, we call the system "almost linear" or "locally linear" at $\vec{p}$.

Def Suppose $\vec{x}' = \vec{F}(\vec{x})$ has the form

$$(\star) \qquad \vec{x}' = A\vec{x} + \vec{g}(\vec{x})$$

and $\vec{x} = \vec{0}$ is an isolated critical pt and $\det A \neq 0$. Then if $\vec{g}(\vec{x})$ has continuous partials and

$$\lim_{\vec{x} \to \vec{0}} \frac{\|\vec{g}(\vec{x})\|}{\|\vec{x}\|} = 0,$$

we say $(\star)$ is "almost linear" at $\vec{x} = \vec{0}$.

Notes ① $\vec{g}(\vec{x})$ must be small near $\vec{0}$ compared to $\vec{x}$.

② Can check $\vec{g}(\vec{x})$ component-wise:

$$\vec{g}(\vec{x}) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}, \quad \lim_{x,y \to 0} \frac{g_i(x,y)}{r} = 0$$

where $i = 1, 2$, and $r = \sqrt{x^2+y^2}$

ex. For F, G polynomials and $\vec{0}$ already critical, ~~this~~ ~~approach~~ finding A is easy:

Let
$$\left.\begin{aligned} \dot{x} &= x - x^2 - xy \\ \dot{y} &= -y - y^2 + 2xy \end{aligned}\right\} \quad \vec{x}' = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{A} \vec{x} + \underbrace{\begin{bmatrix} -x^2 - xy \\ -y^2 + 2xy \end{bmatrix}}_{\vec{g}(\vec{x})}$$

Here, eigenvalues $r_1 = 1, r_2 = -1$ indicate a saddle at origin.