In preparation for the study of non-linear systems, Section 9.1 is a review of the linear systems theory, but given solely in terms of understanding how to characterize the type and stability of the equilibrium solution at the origin of the phase plane.

General facts

- Solutions to $\vec{x}' = A_{2\times 2}\vec{x}$ are typically constructed from exponentials $\vec{x}(t) = \vec{v}e^{rt}$ where $r, \vec{v}$ are eigenvalue/eigenvector pairs of A. Here $\vec{v}$ satisfies $(A - rI)\vec{v} = \vec{0}$ for r a solution to the characteristic equation of A: $\det(A - rI) = 0$.

General facts cont'd.

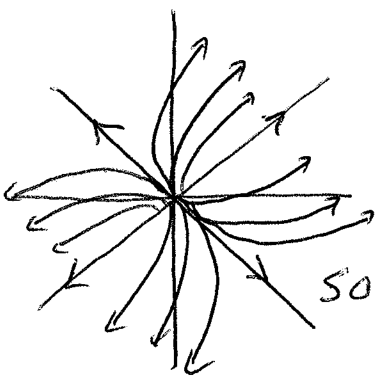
- If $r=0$ is not a solution to $\det(A-rI)=0$, then $\vec{x}=\begin{bmatrix}0\\0\end{bmatrix}$ is the only equilibrium solution of the ODE
- Solutions are integral curves, parameterized by t, in the x_1x_2-plane (phase plane) and a representative sample of curves is called a phase portrait.
- The general shape of the phase portrait determines the <u>type</u> of the equilibrium at $\begin{bmatrix}0\\0\end{bmatrix}$ and the direction of travel along solutions determines the stability of $\begin{bmatrix}0\\0\end{bmatrix}$.
- Both type and stability of $\begin{bmatrix}0\\0\end{bmatrix}$ are fully determined by the eigenvalues of A alone.

General Facts cont'd.

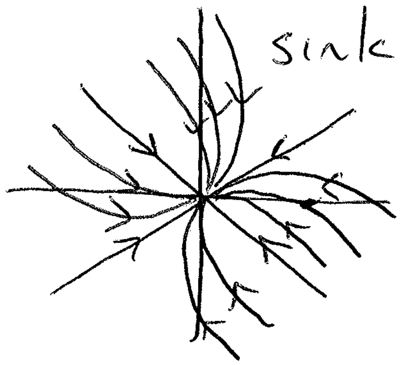
- Classification of $\begin{bmatrix}0\\0\end{bmatrix}$ for $\vec{x}' = A_{2\times 2}\vec{x}$: $r_1, r_2 \neq 0$,

Case I : $r_1 \neq r_2$ real

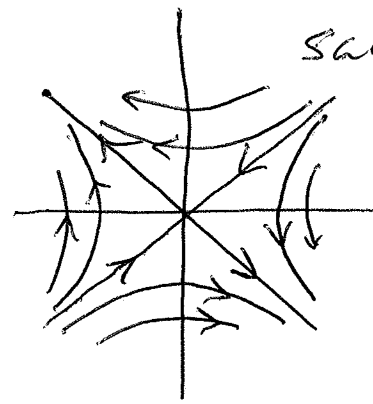
Here, origin is a
- sink $r_1<0, r_2<0$ asympt. stable
- source $r_1>0, r_2>0$ unstable
- saddle $r_1<0, r_2>0$, or $r_1>0, r_2<0$. also unstable



source



sink



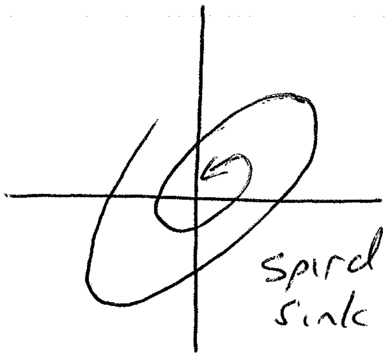
saddle

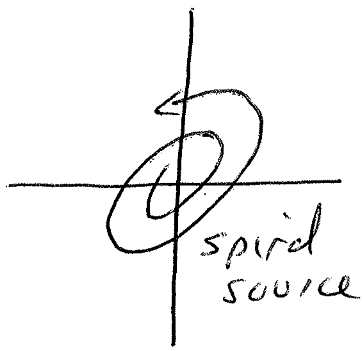
Notice that both a source and a saddle are unstable. But there is a difference. Can you characterize it?

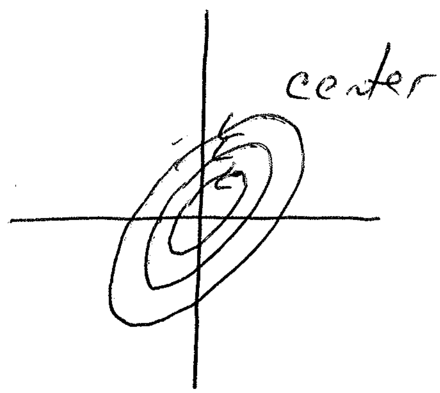
Case II: $r = \lambda \pm i\mu$, $\mu \neq 0$

Eigenvalues are complex conjugates

Here, origin is a
- spiral sink $\lambda < 0$ asympt. stable
- spiral source $\lambda > 0$ unstable
- center $\lambda = 0$ stable (not asymptotically stable)

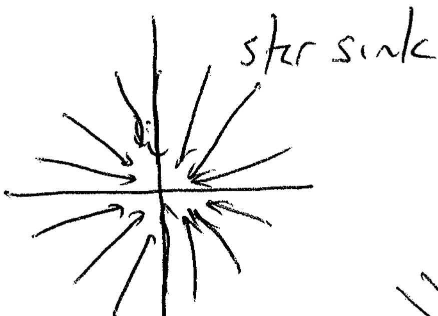




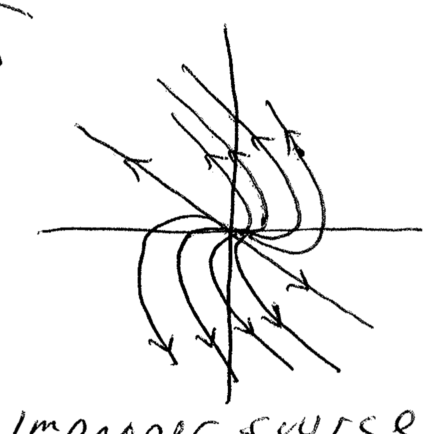


Case III $r_1 = r_2$ (must be real)

Here, origin is a
- star node $r > 0$ unstable, $r < 0$ asympt stable if enough eigenvectors to construct 2 solutions (lin. ind.).



- improper node if there are not enough eigenvectors to construct 2 lin. indep. solutions.



Many 2x2 systems of 1st order homogenous ODEs are not linear (in the dependent variables)

$$(*) \quad \begin{aligned} \dot{x} &= F(x,y) \\ \dot{y} &= G(x,y) \end{aligned}$$

although still autonomous (t is not explicit in the ODE, even as it always is in the solutions).

In vector form, (*) is written $\vec{x}' = \vec{f}(\vec{x})$,

where $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\vec{f}(\vec{x}) = \begin{bmatrix} F(\vec{x}) \\ G(\vec{x}) \end{bmatrix} = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix}$.

Note: Solutions to (*), written as $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

are sometimes given a different notation like $\vec{\varphi}(t) = \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix}$ when the context encourages it.

ex. We can write any $\vec{x}' = A\vec{x}$ in the notation of (*), for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, as

$$\begin{aligned} F(x,y) &= ax + by \\ G(x,y) &= cx + dy \end{aligned}$$

Note: In a non-linear system like (*), there can be more than one isolated singularity (equilibrium solution). Also the origin need not be an equilibrium.

<u>Def.</u> A pt $\vec{p} \in \mathbb{R}^2$ is a <u>critical pt</u> of the system $\vec{x}' = \overrightarrow{f(\vec{x})}$ if $\overrightarrow{f(\vec{p})} = \vec{0}$. Critical pts are equilibria of the system: $\overrightarrow{\varphi(t)} = \vec{p}$. $\forall t \in \mathbb{R}$.

<u>Note</u>: The stability of a critical pt $\vec{p}$ of $\vec{x}' = \overrightarrow{f(\vec{x})}$ is determined in the same way as that of a linear system; by determining how solutions behave <u>around</u> $\vec{p}$.

Recall the Lotka-Volterra eqns

$$\dot{x} = a_1 x - b_1 xy = F(x,y)$$
$$\dot{y} = -a_2 y + b_2 xy = G(x,y)$$

$a_1, a_2, b_1, b_2 > 0$

ex. For $a_1 = 2$, $a_2 = 3$, $b_1 = 1$, $b_2 = 4$
Find all critical pts.

Solution: Here, ODE system is

$$\dot{x} = 2x - xy = x(2-y) = F(x,y)$$
$$\dot{y} = -3y + 4xy = y(4x-3) = G(x,y)$$

We look for $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ where $F(x,y) = 0 = G(x,y)$.

Here ① The origin is critical. ~~Also~~ To find another, assume $x \neq 0$. Then for $F(x,y) = 0$, we must have $y = 2$. Then if $y = 2$, for $G(x,y) = 0$, we must have $x = \frac{3}{4}$.

So ② $x = \frac{3}{4}$, $y = 2$ is also critical.

These are the only two. ∎

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ex. Find all critical pts of

$$\dot{x} = x - x^2 - xy$$
$$\dot{y} = -y - y^2 + 2xy$$

Note: This is Lotka-Volterra with extra terms.

and discuss stability via the JODE phase ~~de~~ portrait.

Solution: We seek all solutions to system

$$F(x,y) = x - x^2 - xy = x(1-x-y) = 0$$
$$G(x,y) = -y - y^2 + 2xy = y(-1-y-2x) = 0.$$

We do this by cases:

① $x = y = 0$. The origin is critical.

② Let $y=0$ and assume $x \neq 0$. Then $G(x,y)=0$. For $F(x,y)=0$ also, we need $1-x-y=0=1-x$. Here $x=1$.

Hence $x=1$, $y=0$ is critical.

③ Let $x=0$ and assume $y \neq 0$. Then $F(x,y)=0$ and to ensure $G(x,y)=0$ also, we need $-1-y-2x = -1-y-2(0) = 0$. So $y=-1$.

Hence $x=0$, $y=-1$ is critical.

Soln cont'd.

(4) Assume $x \neq 0$ and $y \neq 0$. Then $F(x,y) = 0$ only if $1 - x - y = 0$ or along the line $y = 1 - x$. And $G(x,y) = 0$ only when $-1 - y - 2x = 0$, or along the line $y = 2x - 1$.

Both F and G are 0 where these lines intersect:

$$\left.\begin{aligned} x + y &= 1 \\ 2x - y &= 1 \end{aligned}\right\} \text{ solved by } x = \tfrac{2}{3} \;\; y = \tfrac{1}{3}.$$

Hence $x = \frac{2}{3}$, $y = \frac{1}{3}$ is critical.

What do you see in the phase portrait?

- $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are saddles
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is ~~sink~~ an improper source?
- $\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$ is ~~improper~~ a spiral sink.

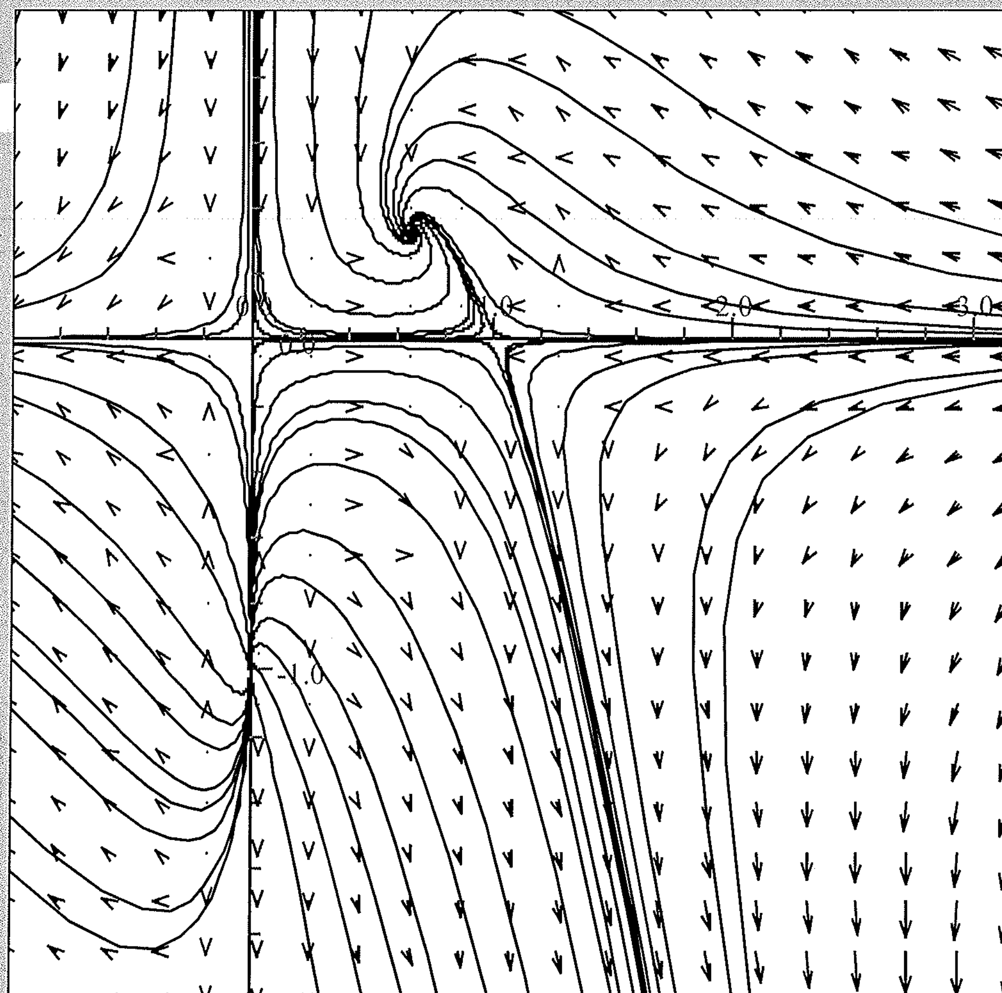
At this point, these are really just guesses.

X

x=-1 **Physical pendulum with friction** x=PI

y=1 y=1

y



t

x

y

y=-2 y=-2

x=-1 x x=PI

eqn #1: dx/dt =	x-x*x-x*y
eqn #2: dy/dt =	-y-y*y+2*x*y

Min. t	-10	Max. t	20
Min. x	-1	Max. x	PI
Min. y	-2	Max. y	1

Num. of segs: t 100 x 20 y 20 Submit All

Show: ☑ Slop ☑ Solutic ☑ Lir ☐ Poi ☑ Ti ☑ A ☐ Init. Cor Mod. Eu Step: 0.1

Add init. cond.: x -0.94361693 y 0.86229508 Submit Show Table Clear All

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2 more definitions

Def. Let the critical pt $\vec{p}$ of $\vec{x}' = \overrightarrow{f(\vec{x})}$ be asymptotically stable (a sink). Then the set

$$B(\vec{p}) = \left\{ \vec{x}^0 \in \mathbb{R}^2 \;\middle|\; \lim_{t \to \infty} \vec{x}(t) = \vec{p} \quad \vec{x}(t_0) = \vec{x}^0 \right\}.$$

is called the basin of attraction of $\vec{p}$ under $\vec{x}' = \overrightarrow{f(\vec{x})}$. It is the set of pts in ~~$\mathbb{R}^2$~~ the phaseplane whose solutions passing through are asymptotic to $\vec{p}$.

Def If a solution $\vec{x}(t)$ to $\vec{x}' = \overrightarrow{f(\vec{x})}$ comprises part of the ~~edge~~ boundary of a basin of attraction, it is called a separatrix.

Note: More broadly, separatrices are ~~non equilibrium~~ noncritical solutions where solution behavior is different on each side.