# 110.302 Lecture 24: 10/28/15

Back to $\vec{x}\,' = A_{2\times 2}\vec{x}$ in the case where the 2 eigenvalues of $A$, $r_1, r_2$ are real and distinct: $r_1 \neq r_2$.

## Some Notes

① This works fine when $0$ is an eigenvalue:

ex. $\vec{x}\,' = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}\vec{x}$. Here char. eqn of $\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$ is $r^2 - 2r = 0$, w/ solns $r_1 = 0$, $r_2 = 2$.

For $r_1 = 0$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a choice of eigenvector.

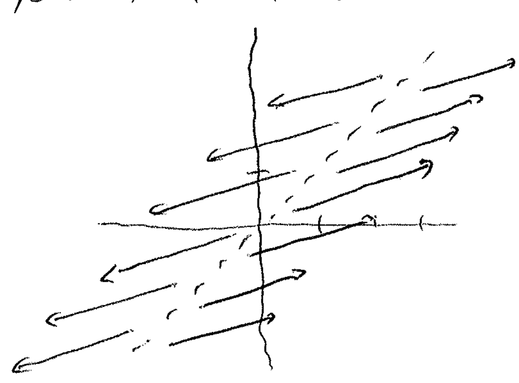
For $r_2 = 2$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is one choice

Then $\vec{x}(t) = c_1\vec{v}_1 e^{r_1 t} + c_2\vec{v}_2 e^{r_2 t}$

$= c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 1 \end{bmatrix}e^{2t}$ is gen. soln.

Q: What does the phase portrait look like here?

Special Note: Here $\det A = 0$ but there are still 2 independent eigenvectors!



(2) This formulation of the general solution still works even for repeated eigenvalues <u>if</u> there are enough eigenvectors.

ex. $\vec{x}' = \begin{bmatrix}1&0\\0&1\end{bmatrix}\vec{x}$. Here characteristic eqn is $r^2-2r+1=0=(r-1)^2$. Hence $r_1=r_2=1$.

The eigenvector eqn $\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=1\begin{bmatrix}x_1\\x_2\end{bmatrix}$

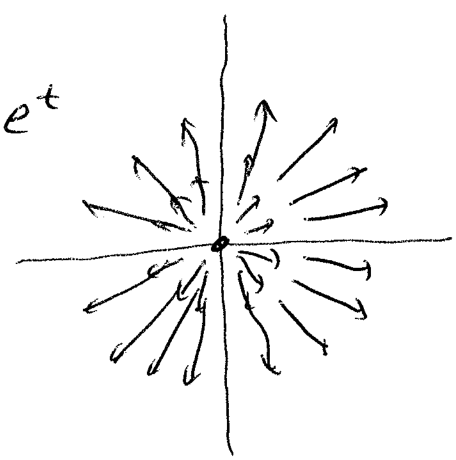
$\Longrightarrow$ $\begin{matrix}x_1=x_1\\x_2=x_2\end{matrix}$ can be solved using 2 lin ind. vectors.

One choice is $\vec{v}_1=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{v}_2=\begin{bmatrix}0\\1\end{bmatrix}$.

Hence

$\vec{x}(t)=c_1\begin{bmatrix}1\\0\end{bmatrix}e^t+c_2\begin{bmatrix}0\\1\end{bmatrix}e^t$

is the general solution.



ex. (~~4~~ 6 pg. 403) $\vec{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$

Here $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\det(A - rI) = \begin{vmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{vmatrix}$

$$= -r \begin{vmatrix} -r & 1 \\ 1 & -r \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 1 & -r \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ -r & 1 \end{vmatrix}$$

$$= -r(r^2 - 1) - 1(-r - 1) + 1(1 + r) = 0$$

$$= -r^3 + r + r + 1 + 1 + r = r^3 - 3r + 2 = 0$$

By long division

$$r^3 - 3r + 2 = (r+1)(r^2 - r - 2) = (r+1)^2(r-2)$$

Hence eigenvalues are $r_1 = 2$, $r_2 = r_3 = -1$

In the book, they solve $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

So general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{-t}$$

exercise: Imagine the 3-dim phase space... What happens near the equilibrium at the origin?

(3) This formulation for finding solutions ~~does not~~ work when there are not enough ~~eigenvectors~~ independent eigenvectors for repeated eigenvalues.

ex. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{x}$. Here, eigenvalues $r_1 = r_2 = -2$.

But the eigenvector eqn $A\vec{v} = -2\vec{v}$, or

$$-2x_1 + x_2 = -2x_1$$
$$-2x_2 = -2x_2$$

is ONLY solved by $x_2 = 0$, $x_1 \neq 0$. ~~An~~ only independent choice would be $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Here, we will need to come up with another method to find another independent soln.

# Properties of phase portraits

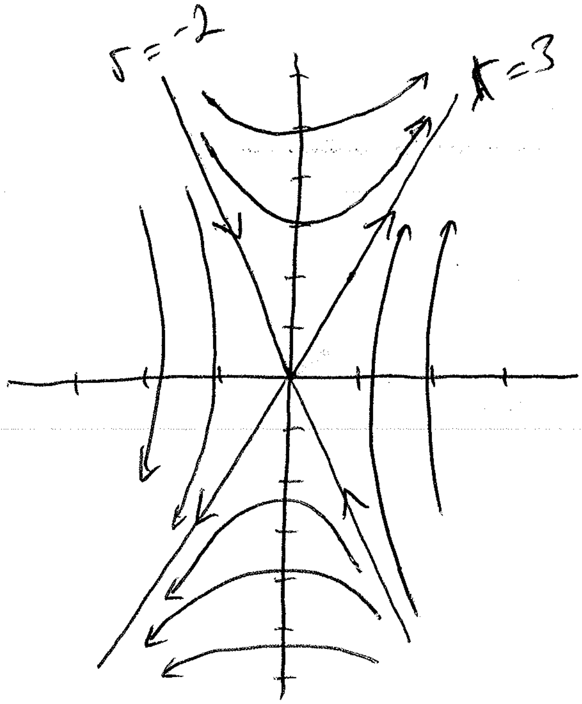
Let $\vec{x}' = A\vec{x}$. We have

① For any $A_{2\times 2}$, the origin is an equilibrium solution ($\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always a solution to $A\vec{x} = \vec{0}$).

② If $\det A \neq 0$, then $\vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the ONLY equilibrium solution (stationary pt, or fixed pt), since $A\vec{x} = \vec{0}$ has $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as the only solution.

③ The eigenvectors of $A$ correspond to lines through the origin where the solutions exhibit straight line motion

Note: These straight lines contain many distinct solutions.




phase portrait for
$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$.
w/ solns
$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

4. The sign of each eigenvalue determines direction along lines (toward origin if $<0$ outward if $>0$).

5. If eigenvalues are real and distinct then these are the only straight lines (why?)

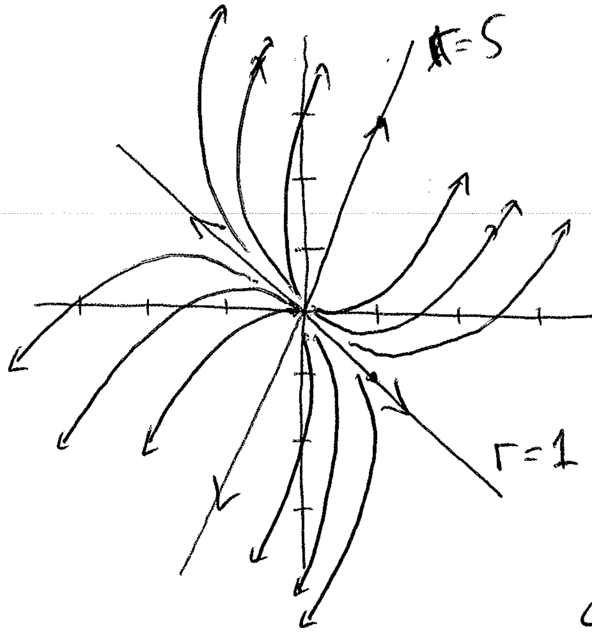
6. Signs of eigenvalues determine "stability" of equilibrium at origin

(How solutions behave "near" the origin. Do they stay nearby, converge to, or diverge from....)

7. Above, origin is called a "saddle pt".
Would you consider it stable?

ex. $\vec{x}' = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$. Here $r_1 = 5$, $r_2 = 1$,

$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.




The general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$$

All solutions tend to origin as $t \to -\infty$ and are unbounded as $t \to \infty$

Origin is called a <u>source</u> and is unstable.

Q: Why is motion curved like above? How to determine?

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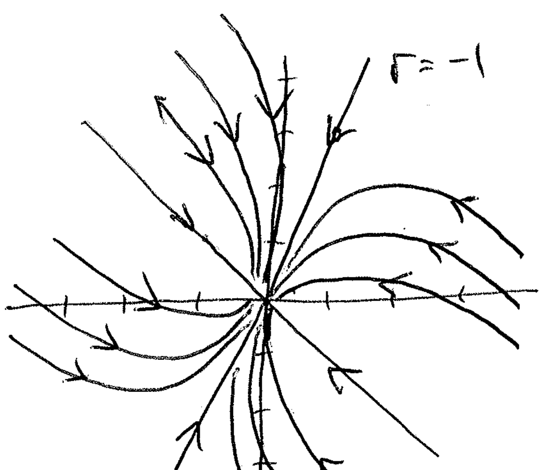
ex. $\vec{x} = \begin{bmatrix} -4 & 1 \\ 3 & -2 \end{bmatrix} \vec{x}$ Here $r_1 = -5$, $r_2 = -1$, with

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$




General solution is

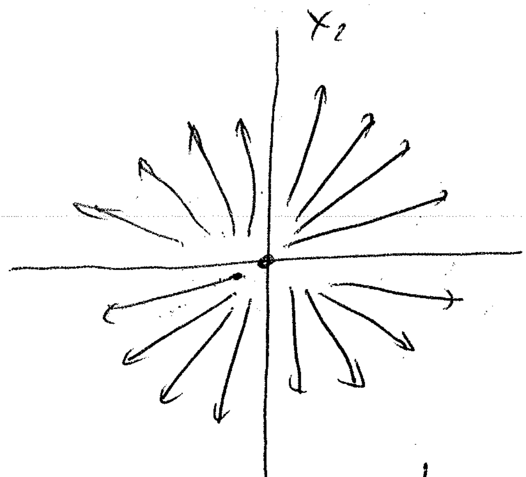
$$\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

Phase portrait is similar to above but different: how?

Here origin is a "sink" and asymptotically stable.

ex. $\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$. Here, as we have seen

$r_1 = r_2 = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$




and $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t$

$= \left( c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^t$

This is another source at the origin here called a <u>star node</u>.

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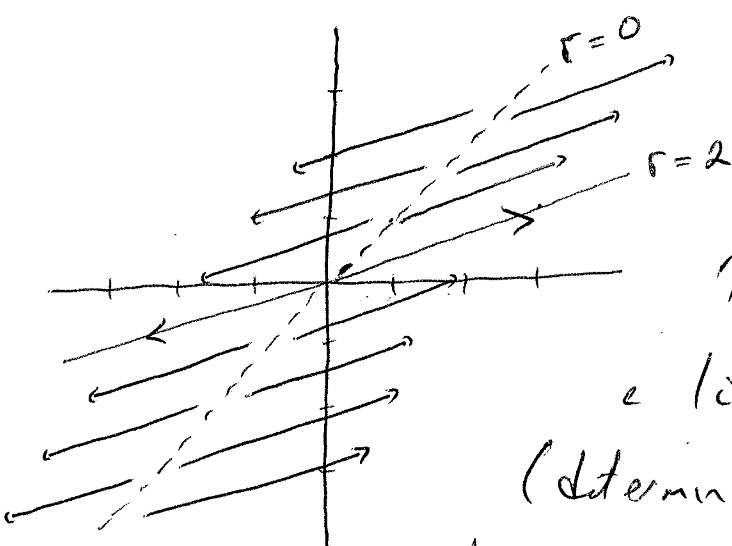
ex. $\vec{x}' = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \vec{x}$. Here $r_1 = 0$, $r_2 = 2$,

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.




General soln is

$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$

This one is special: There is a line of equilibrium solution (determinant is 0). Off the dotted line, motion is straight along lines parallel to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and moving out from dotted line.

Q: What is the stability?