110.302 Lecture 23: 10/26/15

For $\vec{x}\,' = A_{2\times 2}\vec{x}$,

(I) One can construct a slope field in $\mathbb{R}^2$ via matrix multiplication: For $\vec{x} \in \mathbb{R}^2$, the tangent to the solution curve passing through $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is $A\vec{x}$

(a) $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x}\,' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

(b) $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

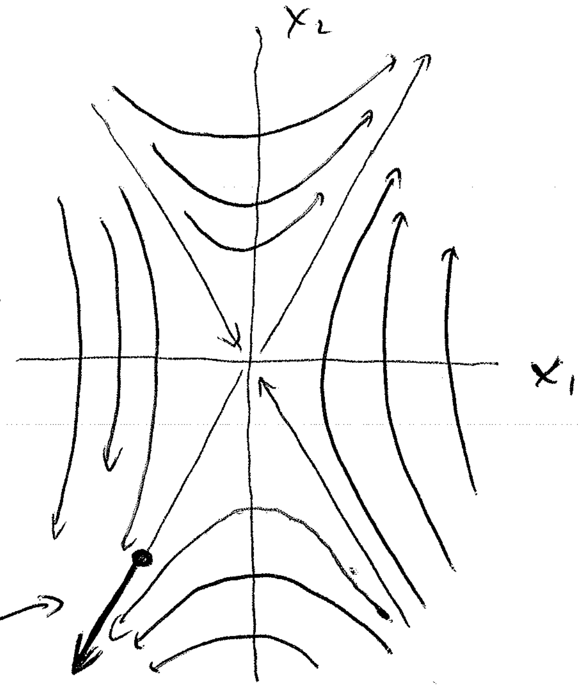
(c) $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

JODE 2D calculator (or similar) is helpful here

(II) Solution curves are integral curves of the slope field:

(a) Given $c_1, c_2 \in \mathbb{R}$, the curve
$$\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} 1 \\ -3 \end{bmatrix}e^{-2t}$$
is one of these curves.

(b) Straight-line motion only occurs when one of c_1, c_2 is 0:



Let $c_1 = -2$, $c_2 = 0$.

$$\Rightarrow \vec{x}(t) = -2\begin{bmatrix}1\\2\end{bmatrix}e^{3t} = \begin{bmatrix}-2\\-4\end{bmatrix}e^{3t}$$

(c) A copy of $\mathbb{R}^2$ with enough representative curves on it to give a good sense of solutions is called a <u>phase portrait</u>:

(d) Solutions are called <u>trajectories</u> or <u>orbits</u>.

(e) General long term behavior of trajectories can be read off easily from a phase portrait:

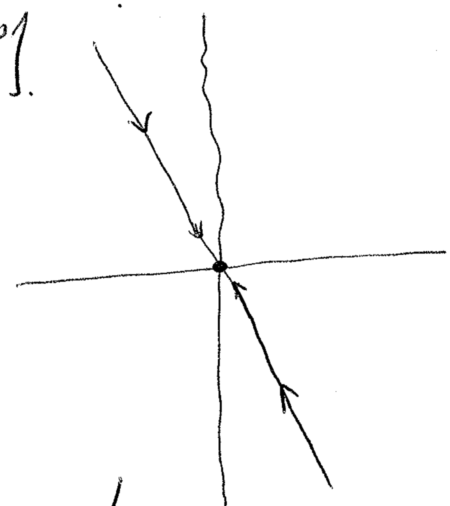
Given $\vec{x}(t) = c_1\begin{bmatrix}1\\2\end{bmatrix}e^{3t} + c_2\begin{bmatrix}1\\-3\end{bmatrix}e^{-2t}$, II e cont'd.

- If $c_1 = 0$, $c_2 \neq 0$, then $\vec{x}(t) = c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

  and $\lim_{t\to\infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

  How about $\lim_{t\to-\infty} \vec{x}(t)$?

  We say it is <u>unbounded</u>.



Note: Solutions never touch nor cross lines.

So since $\vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium soln,

no other solution actually reaches $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

- If $c_1 \neq 0$, $c_2 = 0$?

  $\lim_{t\to\infty} \vec{x}(t)$ DNE or traj is unbounded.

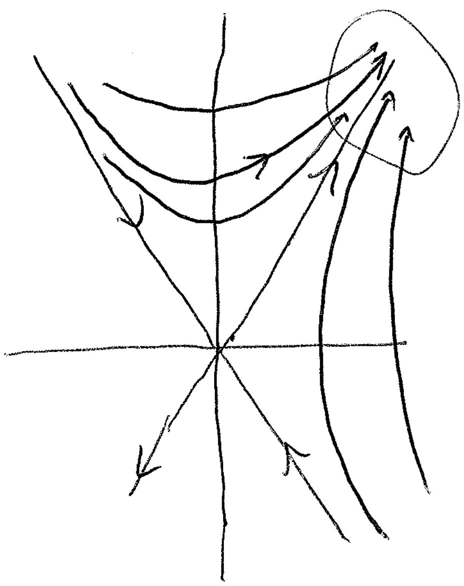
  $\lim_{t\to-\infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

- If $c_1 \neq 0$, and $c_2 \neq 0$? These are the curved lines in the portrait. For these, what can we say about the forward trajectory ($\lim_{t\to\infty} \vec{x}(t)$), or the backward one ($\lim_{t\to-\infty} \vec{x}(t)$)?

One answer? Given $\bar{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

and $c_1 \neq 0$, $c_2 \neq 0$, then as t gets large

$(t \to \infty)$, $\bar{x}(t)$ looks more and more like



$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$, and less and less like $c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

How about in backward time?

(III) Back to solution building via the properties of A: For $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$,

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

the eigenvalues of A ($r_1 = 3$, $r_2 = -2$), and representative eigenvectors ($\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$) are explicitly part of the solution.

Why?

Here, the eigenvalues and eigenvectors of a matrix A satisfy $A\vec{v} = \lambda\vec{v}$, or

$$(A - \lambda I)\vec{v} = \vec{0}.$$

For this to have nontrivial solutions for $\vec{v}$,

- λ must be an eigenvalue, and
- $\det(A - \lambda I)\,\vec{v} = \vec{0}$.

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$$= \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

This is called the <u>characteristic</u> <u>eqn</u> of A, and solutions can be

1. real, distinct
2. real and repeated
3. complex conjugates.

How to relate to solutions of $\vec{x}' = A\vec{x}$?

- Recall $\dot{x} = ax$ is solved by $x(t) = ce^{at}$
- Assume $\dot{\vec{x}} = A\vec{x}$ is also solved by exponentials. For $n=2$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

  Assume $x_1(t) = c_1 e^{rt}$, $x_2(t) = c_2 e^{rt}$

$\Rightarrow$ $\vec{x}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{rt}$, and $\vec{x}' = r \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{rt}$

and then $\vec{x}' = A\vec{x}$ is $r \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{rt} = A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{rt}$

or $\quad r \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, or $r\vec{v} = A\vec{v}$.

And what do solutions to this look like??

r is an eigenvalue

$\vec{v}$ is an eigenvector of r.

Notes ① This works equally well for $n > 2$.

② In the case where $A_{n \times n}$ is real and symmetric (i.e. when $a_{ij} = a_{ji}$)

$\Rightarrow$ all eigenvalues are real and even if repeated, there is a full set of eigenvectors.

We have the following:

For $\dot{\vec{x}} = A_{n\times n}\vec{x}$, when all eigenvalues of A are real and distinct, then

$$\vec{x}(t) = c_1\vec{v}_1 e^{r_1 t} + \cdots + c_n \vec{v}_n e^{r_n t}$$

is the general soln, where $\vec{v}_i$ is ~~the~~ an eigenvector of r_i, for A.