

110.302 Lecture 21: 10/21/15

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We have already seen that  $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is an

eigenvector of the eigenvalue 3, for  $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$ .

What is an eigenvector for  $\lambda = -2$ ?

Back to  $A\vec{x} = \lambda\vec{x}$ :

$$\begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left. \begin{array}{l} x_1 + x_2 = -2x_1 \\ 6x_1 = -2x_2 \end{array} \right\} \Rightarrow \begin{array}{l} 3x_1 = -x_2 \\ 6x_1 = -2x_2 \end{array}$$

Notice how this system is degenerate (has tons of solutions!) It always is...

Choose  $x_1 = 1$  (any nonzero choice for  $x_1$  will do)

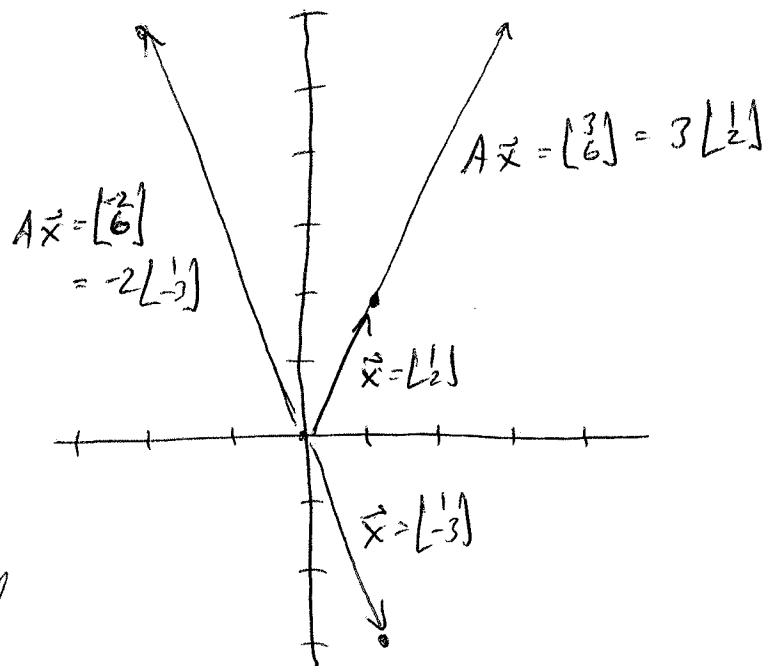
$$\Rightarrow x_2 = -3, \text{ so}$$

$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$  is an eigenvector

for  $\lambda = -2$ . We say

these 2 directions are  
invariant under  $A$ .

They are the only 2 such  
directions.



Back to ODEs:

If  $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  is a vector of variables (fncs of time)

Then  $\vec{x}' = \begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$  is its derivative, and

$$x_1' = p_{11}(t)x_1 + \dots + p_{1n}(t)x_n + g_1(t)$$

⋮

$$x_n' = p_{n1}(t)x_1 + \dots + p_{nn}(t)x_n + g_n(t)$$

is a linear system with matrix form

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

A solution is a vector of fncs  $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

and a set of solutions (if more than one) is

$$\vec{x}^{(1)}(t) = \begin{bmatrix} x_1^{(1)}(t) \\ \vdots \\ x_n^{(1)}(t) \end{bmatrix}, \dots, \vec{x}^{(k)}(t) = \begin{bmatrix} x_1^{(k)}(t) \\ \vdots \\ x_n^{(k)}(t) \end{bmatrix}.$$

Note the notational confusion. We will not deal with  $n$ th order systems, so the context is ok.

Some facts - Let  $\vec{x}' = P(t)\vec{x}$  be a homogeneous linear system ( $\vec{g}(t) = \vec{0}$ ).

① Superposition holds: if  $\vec{x}^{(1)}(t)$  and  $\vec{x}^{(2)}(t)$  both solve  $\vec{x}' = P(t)\vec{x}$ , then so does

$$c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)$$

for any choice of  $c_1, c_2 \in \mathbb{R}$ .

(Any linear comb. of solutions is a solution!)

ex.  $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$ . Here  $P(t) = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$  are constants.

$$\text{Verify that } ① \vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}$$

" is a solution.

$$② \vec{x}^{(2)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix}$$

" also is a solution.

$$\text{Hence so " } c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

$$\text{for } ②, \vec{x}' = \frac{d}{dt} \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{-2t} - 3e^{-2t} \\ 6e^{-2t} + 0(-3e^{-2t}) \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 6x_1 \end{bmatrix}$$

② In  $\mathbb{R}^n$ , there can be at most  $n$  linearly independent vectors. Think

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

as an example.

It is reasonable to conclude that there can be up to  $n$  linearly independent solutions to

$$\vec{x}' = P(t) \vec{x}$$

$$\vec{x}^{(1)}(t) = \begin{bmatrix} x_1^{(1)}(t) \\ \vdots \\ x_n^{(1)}(t) \end{bmatrix}, \dots, \vec{x}^{(n)}(t) = \begin{bmatrix} x_1^{(n)}(t) \\ \vdots \\ x_n^{(n)}(t) \end{bmatrix}$$

They will be independent on some interval  $I$  if for  $t \in I$ ,

$$c_1 \vec{x}^{(1)}(t) + \dots + c_n \vec{x}^{(n)}(t) = \vec{0}$$

can only be solved by  $c_1 = \dots = c_n = 0$ .

Combine all of these vector solutions as columns in a single  $n \times n$  matrix:

$$X(t) = [\vec{x}^{(1)} \dots \vec{x}^{(n)}] = \begin{bmatrix} x_1^{(1)} & \dots & x_1^{(n)} \\ \vdots & \ddots & \vdots \\ x_n^{(1)} & \dots & x_n^{(n)} \end{bmatrix}$$

Then  $\det X \neq 0$  if all columns are independent

Def Let  $X(t)$  is called the Wronskian (determinant) of the solution set, and denoted  $W(\vec{x}^{(1)}, \dots, \vec{x}^{(n)})$ .

Thm If  $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$  are all solutions to  $\vec{x}' = P(t)\vec{x}$  on  $I = (a, b) \subset \mathbb{R}$ , then for all  $t \in I$ , either  $W(\vec{x}^{(1)}, \dots, \vec{x}^{(n)})$  is identically 0 or is never 0.

Def. If  $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$  are all solutions to the  $n$ -dim  $\vec{x}' = P(t)\vec{x}$  on  $I$ , and  $W(\vec{x}^{(1)}, \dots, \vec{x}^{(n)}) \neq 0$  on  $I$  then

(a)  $X(t)$  is called a fund. set of solutions and

(b)  $\vec{\varphi}(t) = c_1 \vec{x}^{(1)} + \dots + c_n \vec{x}^{(n)}$  is the general solution.

ex. Given  $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} \vec{x}$ , with solutions

$$\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix}, \quad \vec{x}^{(2)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} = \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix}$$

$$\text{Since } W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{vmatrix} = -3e^{3t}e^{-2t} - 2e^{3t}e^{-2t} = -5e^t \neq 0 \text{ on } \mathbb{R},$$

we have

$$X(t) = \begin{bmatrix} e^{3t} & e^{-2t} \\ 2e^{3t} & -3e^{-2t} \end{bmatrix}$$

Fundamental set of solutions

and

$$\vec{\varphi}(t) = c_1 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \\ -3e^{-2t} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

is the general solution