Some Linear Algebra

A linear system of equations looks like

$$\underbrace{n \text{ eqns}}\left\{\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ \underbrace{a_{n1}x_1 + \cdots + a_{nn}x_n = b_n}_{n\text{-unknowns}} \end{array}\right.$$

We can write this as a single (matrix) eqn by collecting up the constituent parts into arrays

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A_{n\times n}} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}_{n\times 1}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}}_{\vec{b}_{n\times 1}}$$

Here $b_2 = (\text{row } 2 \text{ of } A) \cdot \vec{x}$ where the dot is matrix multiplication.

Some facts about matrices and matrix eqns.

① If $\vec{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ in $A\vec{x} = \vec{b}$, the eqn is called homogeneous.

② A solution to $A\vec{x} = \vec{b}$ is a choice of $\vec{x}$ which satisfies the eqn.

③ If $\det A \neq 0$, the system has a unique soln.

④ If $A\vec{x} = \vec{0}$ and $\det A \neq 0$, then $\vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ is the only solution.

If $\det A = 0$, then tons of solutions

($A\vec{x} = \vec{0}$ is never inconsistent) (why not?)

⑤ If $\det A \neq 0$, then the inverse matrix of A, A^{-1}, exists and can be used to "solve" $A\vec{x} = \vec{b}$:

$$A\vec{x} = \vec{b} \implies \underbrace{A^{-1} \cdot A}\, \vec{x} = A^{-1}\vec{b}$$

$$I_n \vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

Here

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \text{n-dim Identity matrix}$$

⑥ The idea of solving a system of eqns involves adding multiples of eqns to other eqns in order to produce new simpler eqns.

In matrices, these are the elementary row operations one performs to A to reduce the number of non-zero entries. But what one does to A, one must also do to $\vec{b}$.

Hence to solve $A\vec{x}=\vec{b}$, one works with the augmentation matrix

$$A|\vec{b} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right]$$

All relevant info. about $A\vec{x}=\vec{b}$ is encoded in $A|\vec{b}$.

⑦ All vectors, by convention, are considered column vectors. To talk about a row vector,

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{x}^T = [x_1 \cdots x_n]$$

one should either specify "row vector", or take the transpose of a column vector.

<u>Def.</u> A set of vectors $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ (careful of this notation) of the same size are said to be <u>linearly dependent</u> (on each other) if $\exists$ real numbers $c_1, \ldots, c_n \in \mathbb{R}$, not all 0, where

$$c_1 \vec{x}^{(1)} + \cdots + c_n \vec{x}^{(n)} = 0$$

Otherwise they are <u>linearly independent</u>.

<u>Note:</u> The columns of $A_{n\times n}$ are <u>linearly independent</u> iff $\det A \neq 0$.

(8) For $A\vec{x}=\vec{b}$, think of $\vec{x}, \vec{b} \in \mathbb{R}^n$
where $\mathbb{R}^n$ = the set of all n-vectors }.

Then an $n\times n$ matrix $A_{n\times n}$ can be considered a linear transformation of $\mathbb{R}^n$ (a function taking $\mathbb{R}^n$ to $\mathbb{R}^n$) taking $\vec{x}$ to $\vec{b} = A\vec{x}$:

$$A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\vec{x} \overset{A}{\longmapsto} \vec{b} = A\vec{x}$$

A takes n-vectors to n-vectors, where $\vec{b}$ is the image of $\vec{x}$ under A.

ex. Let $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$. Then $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

and $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$$= 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Notice that in the last case,
the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is mapped to a multiple of itself.
This is <u>special</u>!

(9) There is a special eqn in linear algebra:

$$A\vec{x} = \lambda\vec{x}$$

$A_{n\times n}$ - matrix
$\vec{x}$ - n-vector
λ - scalar

A choice of $\bar{x}$ and λ which satisfy this equation indicate a direction (of $\vec{x}$) unchanged via multiplication by A, and expanded or contracted by a factor λ.

Here $\vec{x}$ is called an <u>eigenvector</u> of A, and λ is its corresponding <u>eigenvalue.</u>

ex. In the above example, the vector $\vec{x} = \begin{bmatrix}1\\2\end{bmatrix}$ is an <u>eigenvector</u> of $A = \begin{bmatrix}1 & 1\\6 & 0\end{bmatrix}$ with corresponding eigenvalue 3.

(2) Any multiple of $\begin{bmatrix}1\\2\end{bmatrix}$ is also an eigenvector of A corresponding to 3.

(b) There is another eigenvalue of A: (-2).
How to find these?

The equation $A\vec{x} = \lambda\vec{x}$ has lots of solutions
($n+1$ unknowns and only n equations).

But the values of λ are rare. To find them
rewrite $A\vec{x} = \lambda\vec{x}$:

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0} \qquad \text{place everything on one side}$$

$$A\vec{x} - \lambda I_n\vec{x} = \vec{0} \qquad \text{make coefficient of } \vec{x} \text{ a matrix}$$

$$(A - \lambda I_n)\vec{x} = \vec{0} \qquad \text{Create a homogeneous system}$$

The only way nontrivial solutions exist is if
$\det(A - \lambda I_n) = 0$. But this equation only
has λ in it!!

ex. ~~det(A − λI₂)~~ = Let $A = \begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix}$. Then

$$\det(A - \lambda I_2) = 0 = \det\left(\begin{bmatrix} 1 & 1 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 1 \\ 6 & -\lambda \end{bmatrix}$$

$$= (1-\lambda)(-\lambda) - 6 = 0 = \lambda^2 - \lambda - 6 = (\lambda-3)(\lambda+2)$$

Here eigenvalues are $\lambda = 3$, $\lambda = -2$.