110.302 Lecture 19: 10/15/15

Consider the "system" of 2 1^{st} order ODEs in 2 variables:

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 xy \\ \frac{dy}{dt} &= -a_2 y + b_2 xy, \end{aligned}\right\} \quad a_1, a_2, b_1, b_2 > 0 \text{ constants.}$$

Here, both $x(t)$, $y(t)$ are fncs of time, and their evolution (derivatives) are intertwined (coupled).

Many applications appear this way. These are called the Lotka-Volterra equations: model the population size of 2 species in a closed environment (predator and prey)

A solution is a set of expressions for $x(t)$ and $y(t)$ that satisfy both equations.

Q: Say $x(t)$ and $y(t)$ represent rabbits and foxes (not necessarily respectively). Can you tell from the ODE system which is which? How?

Why study systems of coupled equations?

2 reasons:

① Many apps appear this way. There are many ~~de~~ measurable quantities all depending on a single independent variable (not like vector calculus). In general, this looks like

$$(*) \qquad \left.\begin{aligned} \dot{x}_1 &= F_1(\cancel{t}, x_1, \cdots, x_n) \\ \dot{x}_2 &= F_2(t, x_2, \cdots, x_n) \\ &\vdots \\ \dot{x}_n &= F_n(t, x_1, \cdots, x_n) \end{aligned}\right\} \text{1st order system of ODEs.}$$

where $x_1, \cdots, x_n$ are the set of n dependent variables and time t is the independent var.

② <u>Any</u> higher order ODE can be transformed (rewritten) as a system of 1st order ODEs:

Let $\quad y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$.

Given the new vars:

$$x_1 = y, \; x_2 = y', \; x_3 = y'', \; \cdots, \; x_n = y^{(n-1)},$$

we get

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= F(t, x_1, \cdots, x_n) \end{aligned}\right\} \begin{aligned} \dot{x}_1 &= \dot{y} = y' = x_2 \\ \dot{x}_2 &= (y')' = y'' = x_3 \\ &\vdots \\ \dot{x}_n &= (y^{n-1})' = y^{(n)} = F. \end{aligned}$$

A solution to $(*)$ is a set of functions

$$x_1(t), x_2(t), \ldots, x_n(t)$$

If initial values are specified, we would need 1 data pt for each $x_i(t)$, $i = 1, \ldots, n$

$$x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0.$$

This is identical to

$$y(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_0^{(n-1)}$$

n-bits of initial data.

<u>Existence</u> and <u>uniqueness</u> for 1^{st} <u>order</u> systems (similar to that for a single eqn)

<u>Thm</u> In $(*)$, let $F_1, \ldots, F_n$ <u>and</u> all of

$$\frac{\partial F_1}{\partial x_1}, \ldots, \frac{\partial F_1}{\partial x_n}, \frac{\partial F_2}{\partial x_1}, \ldots \frac{\partial F_2}{\partial x_n}, \ldots, \frac{\partial F_n}{\partial x_1}, \ldots, \frac{\partial F_n}{\partial x_n}$$

(all partials wrt dep vars x_i, but not t)

be continuous in a region R of the $(n+1)$-dim $tx_1 \cdots x_n$-space defined by

$$\alpha < t < \beta, \ \alpha_1 < x_1 < \beta_1, \ \ldots, \ \alpha_n < x_n < \beta_n,$$

and let $p = (t_0, x_1^0, \ldots, x_n^0) \in \mathbb{R}^{n+1}$.

$\Rightarrow$ on an interval $|t - t_0| < h$, there is a unique solution to $(*)$ defined on the interval and passing through p.

Note: The proof is similar to that of the 1-dim case.

<u>Def</u> (*) is linear if each F_i is linear in all of the x_i's, $i=1,\dots,n$. Indeed, if

$$(A)\quad \begin{cases} x_1' = p_{11}(t)\,x_1 + \cdots + p_{1n}(t)\,x_n + g_1(t) \\ x_2' = p_{21}(t)\,x_1 + \cdots + p_{2n}(t)\,x_n + g_2(t) \\ \quad\vdots \\ x_n' = p_{n1}(t)\,x_1 + \cdots + p_{nn}(t)\,x_n + g_n(t) \end{cases}$$

and <u>homogeneous</u> if each $g_i(t) \equiv 0$.

Note: Solutions exist <u>and</u> are unique for a linear system of ODEs (like (A)) on an interval I if <u>all</u> $p_{ij}(t)$ and $g_i(t)$ are continuous on I.