110.302 Lecture 18: 10/14/15

Q: How to solve the nth-order linear ODE?

$$\underbrace{a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y}_{L[y]} = G(t)$$

or

$$\overbrace{y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y}^{L[y]} = g(t)$$

A: Same as before is the short answer:

The homogeneous part ($L[y]=0$), if the coefficients are constants, can be solved by exponentials: Assume $L[e^{rt}]=0$ to construct the characteristic eqn:

$$(*) \quad a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$$

The roots of (*) correspond to solutions $y(t)=e^{rt}$ which are solutions to ~~the~~ $L[y]=0$.

The rest of the theory holds also:

① If roots of (*) can be found (all of them, counting multiplicity and complex conjugates, one can construct an n-parameter family of solutions (the fund. set of solns: then ex. Suppose (*) has all real distinct roots

$r_1 \neq r_2 \neq \cdots \neq r_n : \mathbb{R}e\,n$

$$y(t) = c_1 e^{r_1 t} + \cdots + c_n e^{r_n t}$$

is the general solution.

ex. For repeated roots, the pattern is similar to the 2nd order version

ⓐ Suppose characteristic eqn of a 5th order ODE were $(r-2)(r+1)^3(r-5) = 0$

then $r_1 = 2$, $r_2 = r_3 = r_4 = -1$, $r_5 = 5$, and

$$y(t) = c_1 e^{2t} + c_2 e^{-t} + c_3 t e^{-t} + c_4 t^2 e^{-t} + c_5 e^{5t}$$

ⓑ Suppose $(r^2-6)(r^2-4r+13)^2 = 0$

$\Rightarrow r_1 = \sqrt{6}$, $r_2 = -\sqrt{6}$, $r_3 = r_5 = 2+3i$

$r_4 = r_6 = 2-3i$

and $y(t) = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} + e^{2t}(c_3 \cos 3t + c_4 \sin 3t) + t e^{2t}(c_5 \cos 3t + c_6 \sin 3t)$

Solution methods for non-homogeneous linear nth order ODEs are the same:

① Undetermined Coefficients - exactly the same as the 2nd order version

② Variation of Parameters

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Assume $Y(t) = u_1 y_1 + \cdots + u_n y_n$, for $y_1, \cdots, y_n$ solutions to the homogeneous version.

Playing the same game by taking derivatives, making assumptions (to simplify) and plugging into the ODE, one obtains a set of n equations

$$
\begin{aligned}
u_1' y_1 + \cdots + u_n' y_n &= 0 \\
u_1' y_1' + \cdots + u_n' y_n' &= 0 \\
u_1' y_1'' + \cdots + u_n' y_n'' &= 0 \\
\vdots \qquad \ddots \qquad &\vdots \\
u_1' y^{(n-1)} + \cdots + u_n' y_n^{(n-1)} &= g(t)
\end{aligned}
$$

This set of n-equations in n-unknowns (the derivatives of the u_i's) can be solved, and solution will be unique (if $W(y_1, \cdots, y_n) \neq 0$).

And lastly, Reduction of Order methods work perfectly well:

Assume $y_1(t)$ solves an nth order linear homog ODE. Then $y_2(t) = v(t)\,y_1(t)$ leads to a $(n-1)$th order ODE in v'. ~~[illegible]~~ Not necessarily easier, but perhaps $(n-1)$th order ODE may have obvious solutions?

<u>exercise</u>: Derive the 2nd order ODE in v' given $y''' + p y'' + q y' + r y = 0$.

Consider the "system" of 2 1^{st} order ODEs in 2 variables:

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 xy \\ \frac{dy}{dt} &= -a_2 y + b_2 xy, \end{aligned}\right\} \quad \begin{aligned} &a_1, a_2, b_1, b_2 > 0 \\ &\text{constants.} \end{aligned}$$

Here, both $x(t), y(t)$ are fncs of time, and their evolution (derivatives) are intertwined (coupled).

Many applications appear this way. These are called the Lotka-Volterra equations: model the population size of 2 species in a closed environment (predator and prey)

A solution is a set of expressions for $x(t)$ and $y(t)$ that satisfy both equations.

Q: Say $x(t)$ and $y(t)$ represent rabbits and foxes (not necessarily respectively). Can you tell from the ODE system which is which? How?

Why study systems of coupled equations?

2 reasons:

① Many apps appear this way. There are many ~~de~~ measurable quantities all depending on a single independent variable (not like vector calculus). In general, this looks like

$$(*) \qquad \left.\begin{aligned} \dot{x}_1 &= F_1(t, x_1, \cdots, x_n) \\ \dot{x}_2 &= F_2(t, x_2, \cdots, x_n) \\ &\;\vdots \\ \dot{x}_n &= F_n(t, x_1, \cdots, x_n) \end{aligned}\right\} \text{1st order system of ODEs.}$$

where $x_1, \cdots, x_n$ are the set of n dependent variables and time t is the independent var.

② Any higher order ODE can be transformed (rewritten) as a system of 1st order ODEs:

Let $\quad y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$.

Given the new vars:

$$x_1 = y, \; x_2 = y', \; x_3 = y'', \; \cdots, \; x_n = y^{(n-1)},$$

we get

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\;\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= F(t, x_1, \cdots, x_n) \end{aligned}\right\} \quad \begin{aligned} \dot{x}_1 &= \dot{y} = y' = x_2 \\ \dot{x}_2 &= (y')' = y'' = x_3 \\ &\;\vdots \\ \dot{x}_n &= (y^{n-1})' = y^{(n)} = F. \end{aligned}$$