110.302 Lecture 19: 10/9/15

The nth-order version of a linear ODE is

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = G(t)$$

which can also be written like an operator

$$(*) \quad L[y] = y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

where we divide each of the coefficients in the top description by $a_n(t)$ (ex. $p_1(t) = \frac{a_{n-1}(t)}{a_n(t)}$)

The theory generalizes in the obvious ways:

I) If the ODE is an IVP, then we will need n pieces of information to completely determine a solution (think n integrations to get solution creating an n-parameter family of functions:

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots, \quad y^{(n-2)}(t_0) = y_0^{(n-2)}, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

Thm (Existence and Uniqueness)

(II) In (*), if $p_1, \cdots, p_n, g$ are all continuous on some common interval I, then there exists a unique solution to (*) ~~the~~ passing through any set of initial values @ $t_0 \in I$.

(III) Superposition Holds: if $y_1(t)$ and $y_2(t)$ both solve $L[y]=0$ (homogeneous) where $L[y]$ corresponds to the nth order ODE (*), then $c_1 y_1(t) + c_2 y_2(t)$ is also a solution.

(IV) Given n solutions $y_1, \cdots, y_n$ to an nth order homogeneous $L[y]=0$, if

$$W(y_1, \cdots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

Wronskian

is nonzero at $t \in I$, then every solution to the ODE is a linear combination of $y_1, \cdots, y_n$.

In this case, a fundamental set of solutions.

then is $y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$

(V) In fact, it can be shown that for any choice of $y_1, \cdots, y_n$ solns to $(*)$,

$$W(y_1, \cdots, y_n) = c\, e^{-\int p_1(t)\,dt}$$

and is either always 0 on I where $p_1(t)$ is cont, or never 0 on I.

(VI) if in $(*)$, $g(t) \neq 0$, then a general solution is the same as that of a 2nd order nonhomogeneous ODE:

$$y(t) = \underbrace{c_1 y_1(t) + \cdots + c_n y_n(t)}_{\substack{\text{fund set of} \\ \text{solutions to} \\ L[y]=0}} + \underbrace{Y(t)}_{\substack{\text{any particular} \\ \text{solution to} \\ L[y]=g(t)}}$$