<u>Reduction of order</u> – Using one solution to an nth-order ODE to create an (n-1)th order ODE.

Suppose (*) $y'' + p(t)y' + q(t)y = 0$ has $y_1(t)$ as a non-zero solution. If you can find an independent second solution, you are done.

<u>Guess:</u> ~~Any~~ Assume second solution has the form

$$y_2(t) = v(t)y_1(t)$$

for some unknown function $v(t)$.

Why? You will see.

<u>Goal:</u> Try to solve for $v(t)$.

Now if $y_2(t)$ solves the ODE, then

$$y_2'(t) = \frac{d}{dt}[v(t)y_1(t)] = v'(t)y_1(t) + v(t)y_1'(t)$$

$$y_2''(t) = v''(t)y_1(t) + \underbrace{v'(t)y_1'(t) + v'(t)y_1'(t)}_{2v'(t)y_1'(t)} + v(t)y_1''(t)$$

Substitute this back into the original ODE:

$$\underbrace{(v''y_1 + 2v'y_1' + vy'')}_{y_2''} + p\underbrace{(v'y_1 + vy_1')}_{y_2'} + q\underbrace{vy_1}_{y_2} = 0$$

Rearrange in terms of v and derivatives:

$$y_1v'' + (2y_1' + py_1)v' + \underbrace{(y_1'' + py_1' + qy_1)}_{=0 \text{ since } y_1 \text{ solves the ODE}}v = 0$$

(This is why we guess that $y_2 = vy_1$)

We are left with:

(*) $$y_1v'' + (2y_1' + py)v' = 0$$

This is a 2^{nd} order ODE in $v(t)$

But this is a 1^{st}-order ODE in $v'(t)$!!

$\boxed{\text{Solve for } v'(t). \text{ Then integrate to get } v(t).}$

<u>Notes</u> ① Thinking of (★) as a 1^{st} order ODE in $v'(t)$ is called <u>reducing</u> the <u>order</u>.

② If the coefficient of $v(t)$ weren't 0, then we cannot do this! Since the lowest order derivative is v', we can.

③ For independence,

$$W(y_1, v y_1) = \begin{vmatrix} y_1 & v y_1 \\ y_1' & v' y_1 + v y_1' \end{vmatrix}$$

$$= v_1 y_1^2 + \underbrace{v y_1 y_1' - v y_1 y_1'}_{0} = v_1 y_1^2$$

As long as $v_1' \neq 0$, (so $v_1(t)$ is not a constant) $y_2 = v_1 y_1$ will be independent of y_1

ex. $y_1(t) = \frac{1}{t}$ solves $t^2 y'' + 3ty' + y = 0$
on the interval $t > 0$.
Find a fundamental set of solutions.

<u>Strategy</u>: Use reduction of order.

<u>Solution</u>: Both $p(t) = \frac{3}{t}$, $q(t) = \frac{1}{t^2}$ are C^0 on $(0, \infty)$.

Assume $y_2(t) = v(t) y_1(t) = \frac{v(t)}{t}$.

Then ~~...~~ so $v(t)$ solves

$$y_1 v'' + (2y_1' + p y_1) v' = 0, \quad \text{or}$$

$$\frac{1}{t} v'' + \left(2\left(\frac{-1}{t^2}\right) + \left(\frac{3}{t}\right)\left(\frac{1}{t}\right)\right) v' = 0, \quad \text{or}$$

$$\frac{v''}{t} + \frac{v'}{t^2} = 0 \implies t v'' + v' = \frac{d}{dt}[t v'] = 0$$

which implies $tv' = c$, a constant, or $v'(t) = \frac{c}{t}$, or

$v(t) = c_1 \ln t + c_2$ on $t > 0$.

Hence $y_2(t) = \frac{v(t)}{t} = \frac{c_1 \ln t + c_2}{t}$

Questions to ask

① Does $y_2(t)$ actually solve the original ODE?

$$y_2(t) = \frac{c_1 \ln t + c_2}{t}, \quad y_2'(t) = c_1\left(\frac{1-\ln t}{t^2}\right) - \frac{c_2}{t^2}.$$

$$y_2'' = c_1\left(\frac{2\ln t - 3}{t^3}\right) + \frac{2c_2}{t^3}$$

Here
$$t^2y'' + 3ty' + y = 0 = t^2\left(c_1\left(\frac{2\ln t-3}{t^3}\right) - \frac{2c_2}{t^3}\right)$$
$$+ 3t\left(c_1\left(\frac{1-\ln t}{t^2}\right) - \frac{c_2}{t^2}\right)$$
$$+ \frac{c_1 \ln t + c_2}{t}$$
$$= c_1\left(\frac{2\ln t - 3}{t}\right) - \frac{2c_2}{t} + 3c_1\left(\frac{1-\ln t}{t}\right) - \frac{3c_2}{t} + c_1\frac{\ln t}{t} + \frac{c_2}{t} = 0$$

② Notice that $y_1 = \frac{1}{t}$ appears as a summand in $y_2 = c_1\frac{\ln t}{t} + \frac{c_2}{t}$. By superposition, it is not really needed. Check independence of y_1, y_2.

Hence fund. set of soln is

$$y(t) = (\text{constant})\, y_1 + (\text{constant})\, y_2$$
$$= (\text{constant})\frac{1}{t} + (\text{constant})\left(c_1\frac{\ln t}{t} + c_2\left(\frac{1}{t}\right)\right)$$

Combine constants to get

$$\boxed{y(t) = \frac{K_1}{t} + K_2\frac{\ln t}{t}}$$ as the general solution.

<u>Application</u>: Given $ay''+by'+cy=0$.

suppose characteristic eqn has only 1 real soln.

Then $r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = -\dfrac{b}{2a}$

Here $y_1(t) = e^{-\frac{b}{2a}t}$ solves the ODE, but this is the only exponential function that does.

To find another function, <u>reduce the order</u>:

Assume $y_2 = v(t)e^{-b/2a\,t}$, where $v(t)$ solves

$$v''y_1 + (2y_1' + py_1)v' = 0, \text{ or}$$

Note: Here $p(t) = \frac{b}{a}$

$$e^{-\frac{b}{2a}t}v'' + \underbrace{\left(2\left(\tfrac{-b}{2a}e^{-\frac{b}{2a}t}\right) + \tfrac{b}{a}e^{-\frac{b}{2a}t}\right)}_{0}v' = 0$$

$$\Rightarrow e^{-\frac{b}{2a}t}v'' = 0 \Rightarrow v'' = 0 \Rightarrow v(t) = K_1t + K_2,$$

so $y_2(t) = (K_1t+K_2)e^{-b/2a\,t}$

Exercise: Calculate $W(y_1, y_2)$ here!

Hence $y_1(t)$, $y_2(t)$ form a fund. set of solutions,

$$y(t) = (\text{constant})\, e^{-\frac{b}{2a}t} + (\text{constant})(K_1 t + K_2)\, e^{-\frac{b}{2a}t}$$

$$\boxed{y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}}$$

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ex. Solve $25y'' - 20y' + 4y = 0$, $y(0) = 5$, $y'(0) = \frac{3}{2}$.

Solution: Here discriminant $b^2 - 4ac = 400 - 400 = 0$

Hence $r_1 = r_2 = r = -\frac{b}{2a} = \frac{2}{5}$.

Hence fund. set of solutions is

$$y(t) = c_1 e^{\frac{2}{5}t} + c_2 t e^{\frac{2}{5}t}$$

As for the particular soln:

$$y(0) = c_1 e^0 + c_2 (0) e^0 = 5 = c_1$$

So $y(t) = 5e^{\frac{2}{5}t} + c_2 t e^{\frac{2}{5}t}$

And $y'(t) = 5\left(\frac{2}{5}\right)e^{\frac{2}{5}t} + c_2 e^{\frac{2}{5}t} + \frac{2c_2}{5} t e^{\frac{2}{5}t}\Big|_{t=0}$

$$= 2 + c_2 = \frac{3}{2} \implies c_2 = -\frac{1}{2}.$$

Soln is $\boxed{y(t) = 5e^{\frac{2}{5}t} - \frac{t}{2} e^{\frac{2}{5}t}}$