Back to the constant coefficients case:

$$(*) \qquad ay'' + by' + cy = 0$$

Let $a = 1 = c$, $b = 0$. Here $y'' + y = 0$ has characteristic equation $r^2 + 1 = 0$ (no real solns).

But, we know $y_1(t) = \cos t$, $y_2(t) = \sin t$ solve $y'' + y = 0$ and since $W(\cos t, \sin t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$.

These solns are independent, and

$$\boxed{y(t) = c_1 \cos t + c_2 \sin t}$$

is a fund. set of solns.

Q: How can we get that from the characteristic eqn?

First, $r^2 + 1 = 0$ does have 2 solns: $r = \frac{-0 \pm \sqrt{0^2 - 4}}{2} = \pm i$

Sticking to the exponential theme:

$$y_1(t) = e^{it}, \quad y_2(t) = e^{-it}$$

are 2 solns. (But they are not real)

Recall Euler's Formula

$$e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt)$$

Q: Can we construct real solutions from these?

Suppose an ODE has characteristic eqn

$$ar^2+br+c=0, \quad r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\lambda\pm\mu i, \ \mu\neq 0.$$

HW Note: ~~The~~ 2 complex roots of a real quadratic polynomial must be conjugates. why?

Writing 2 exponential sols

$$y_1(t)=e^{(\lambda+\mu i)t} \qquad y_2(t)=e^{(\lambda-\mu i)t}$$

$$=e^{\lambda t}(\cos\mu t+i\sin\mu t) \qquad =e^{\lambda t}(\cos\mu t-i\sin\mu t)$$

We see they are not real. But the real ODE must real solutions!

By superposition, any linear combination of y_1, y_2 is also a solution:

Hence $\frac{1}{2}(y_1(t)+y_2(t))=\frac{1}{2}(e^{\lambda t}\cos\mu t+i\sin\mu t)+e^{\lambda t}\cos\mu t-i\sin\mu t)$

$=e^{\lambda t}\cos\mu t$ is a real solution

And $\frac{1}{2i}(y_1(t)-y_2(t))=$ do this $=e^{\lambda t}\sin\mu t$ is a solution.

real!!!

lets call them

$$u(t) = e^{\lambda t}\cos\mu t, \quad v(t) = e^{\lambda t}\sin\mu t$$

These are 2 real solns (the real and imaginary parts of the original complex solns).

<u>And</u> since $W(u,v) =$ calculate this $= \mu e^{2\lambda t} \neq 0$ everywhere as long as $\mu \neq 0$ (making the root complex)

Hence the solns are independent;

Hence, in the case of $ay'' + by' + cy = 0$ with characteristic equation roots $r = \lambda \pm i\mu$, $\mu \neq 0$, the fund. set of solutions is

$$y(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$$
$$= e^{\lambda t}(c_1\cos\mu t + c_2\sin\mu t).$$

exercise: Check that this is a soln.

ex. $y'' + y = 0$. Roots of $r^2 + 1 = 0$ are $r = \pm i$

Here $\lambda = 0$, $\mu = 1$.

$$y(t) = e^{0t}(c_1\cos 1t + c_2\sin 1t) = c_1\cos t + c_2\sin t.$$

ex. Solve the IVP $\quad y'' + 4y' + 13y = 0$

$$y(0) = 2, \quad y'(0) = 7.$$

Solution: The characteristic eqn is $r^2 + 4r + 13 = 0$

and $r = \dfrac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i$

Hence fund set of solns is

$$y(t) = e^{-2t}\left(c_1 \cos 3t + c_2 \sin 3t\right)$$

For a particular soln:

$$y(0) = e^{-2(0)}\left(c_1 \cos 3(0) + c_2 \sin 3(0)\right) = \boxed{2 = c_1}$$

$$y'(0) = -2e^{-2t}\left(2\cos 3t + c_2 \sin 3t\right) + e^{-2t}\left(-6\sin 3t + 3c_2 \cos 3t\right)\Big|_{t=0}$$

$$= -4 + 3c_2 = 7 \qquad c_2 = \frac{11}{3}$$

Particular solution is

$$\boxed{y(t) = e^{-2t}\left(2\cos 3t + \frac{11}{3}\sin 3t\right)}$$

Facts Given $ay''+by'+cy=0$ and

its char eqn $ar^2+br+c=0$ ~~with roots~~ with roots r_1, r_2:

(I) if ~~roots~~ $r_1 \neq r_2$ real, fund set of solns

$$y(t)=c_1e^{r_1t}+c_2e^{r_2t}$$

(II) if ~~complex~~ $r_1=\lambda+i\mu \neq \lambda-i\mu=r_2$

$$y(t)=e^{\lambda t}(c_1\cos\mu t+c_2\sin\mu t)$$

(III) if $r_1=r_2=r$ then

$$y(t)=c_1e^{rt}+c_2e^{rt}=(c_1+c_2)e^{rt}=ke^{rt}$$

is only 1 solution. We will need another ~~but~~ but lin. indep one to construct a fund. set of solns.

Q: How