New question: If you found 2 solutions $y_1, y_2$ to $L[y]=0$, do all solutions look like $c_1 y_1 + c_2 y_2$? Can there be others?

To study this, let's "solve" the IVP

$$L[y]=0, \quad y(t_0)=y_0, \quad y'(t_0)=y_0'$$

using the idea of a "general" solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

We set

$$c_1 \overbrace{y_1(t_0)}^{\text{real \#}} + c_2 \overbrace{y_2(t_0)}^{\text{real \#}} = y_0 \qquad (\ast A)$$
$$c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0'$$

Solve this system for $c_1, c_2$ (2 eqns, 2 unknowns)

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$$c_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)} \qquad c_2 = \frac{y_0 y_1'(t_0) - y_0' y_1(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)}$$

Note: Solutions to $(\ast A)$:

| | |
|---|---|
| 1 soln | lines cross |
| 0 solns | lines parallel |
| $\infty$ solns | lines the same. |

Note: The numerators are different for $c_1, c_2$, but the denominators are the same!

Rewrite the denominator as the determinant of a 2x2 matrix whose entries are the coefficients of (~~A~~A):

$$y_1(t_0)\,y_2'(t_0) - y_1'(t_0)\,y_2(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

This comes from writing (~~A~~A) as a matrix equation:

$$\underbrace{\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

Given this matrix equation, ~~if~~ if $\det A \neq 0$, there is a unique solution $(c_1, c_2)$.

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In our case, in the 2 expressions for $c_1, c_2$:

1. If denominator non-zero, then unique solution
2. If ONLY denominator zero, then no solutions
3. If both numerator, denominator zero, tons of solns.

Call $W = W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$

The Wronskian (determinant) of $y_1, y_2$ at $t_0$.

- Tells you about the solutions to the IVP

  $L[y]=0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$.

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<u>Thm 1</u> Suppose $y_1, y_2$ are 2 solutions to $L[y]=0$ and at the initial values $y(t_0)=y_0$, $y_0'(t_0)=y_0'$

$W(y_1,y_2)(t_0) \neq 0$.

$\Longrightarrow \exists\, c_1, c_2 \in \mathbb{R}$ so that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ solves the IVP.

Note: This ensures that $y_1$ and $y_2$ are fundamentally "different" solutions (read: independent)

Q: What does this mean?

<u>Thm 2</u> If $y_1, y_2$ both solve $L[y]=0$, and if $\exists\, t_0$ where $W(y_1,y_2)(t_0) \neq 0$,
$\Longrightarrow$ $y(t) = c_1 y_1 + c_2 y_2$ includes every soln!

<u>pf.</u> Let $\varphi$ be any soln to the IVP near $t_0$, where $W(y_1,y_2)(t_0) \neq 0$. Then by Thm 1, $c_1 y_1(t) + c_2 y_2(t)$ solves the IVP for some choice of $c_1, c_2 \in \mathbb{R}$. But by uniqueness
$\varphi(t) = c_1 y_1(t) + c_2 y_2(t)$. $\blacksquare$

Hence, given $L[y]=0$, if you find any 2 solutions $y_1, y_2$ where Wronskian is somewhere non-zero, then

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

includes <u>all</u> solutions on the entire <u>interval</u> where Wronskian is nonzero.

Called the general solution or the fundamental set of solutions to $L[y]=0$.

ex. Suppose $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ both solve $L[y]=0$. The Wronskian is

$$W(y_1,y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_1+r_2)\, e^{(r_1+r_2)t}$$

Here, as long as $r_1 \neq r_2$, $W(y_1,y_2)(t) \neq 0$ anywhere on $\mathbb{R}$, and

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is a fundamental set of solutions to $L[y]=0$.

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ex. $y_1(t) = \sin t$, $y_2(t) = \cos t$, $W(y_1,y_2)(t) \equiv 1$

ex. $y_1(t) = \sin t$, $y_2(t) = \cos(t - \frac{\pi}{2})$, $W(y_1,y_2)(t) \equiv 0$.

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when $W(y_1,y_2)(t) \neq 0$, we say $y_1, y_2$ are linearly independent (as functions).

<u>Def</u> Two functions $f(x), g(x)$ are called <u>linearly dependent</u> (LD) on some open interval $I$ if there exists 2 constants $k_1, k_2$ not both $0$, where

$$k_1 f(x) + k_2 g(x) = 0$$

$\forall x \in I$. Otherwise, called <u>linearly independent</u> or LI.

<u>Note:</u> ~~Thm~~ If $\exists$ one pt $x \in I$ where 2 fncs are LI, then the functions are LI on $I$.

<u>Extra</u> The Wronskian only really depends on the ODE in a fundamental way:

<u>Thm</u> Given any 2 solutions to $L[y] = 0$, where $p(t), q(t)$ are cont on an open interval $I$, then

$$W(y_1, y_2) = c\, e^{-\int p(t)\,dt}$$

where $c$ depends on $y_1, y_2$ but not on $t$.

Notes ① If $y_1, y_2$ are LD, then $C = 0$.

② If $y_1, y_2$ are LI, then $W \neq 0$ on all of $I$.

③ Proof is quite interesting!

④ LI and nonzero Wronskian are the same thing for ODES.

pf. Since $y_1, y_2$ solve the ODE

$$\text{ⓐ} \quad y_1'' + p(t)\, y_1' + q(t)\, y_1 = 0$$

$$\text{ⓑ} \quad y_2'' + p(t)\, y_2' + q(t)\, y_2 = 0$$

Mult ⓐ by $-y_2$ and ⓑ by $y_1$ and add (eliminating $q(t)$)

$$\underbrace{\underbrace{(y_1 y_2'' - y_2 y_1'')}_{W'(y_1, y_2)} + p(t)\underbrace{(y_1 y_2' - y_2 y_1')}_{W(y_1, y_2)} = 0}$$

$$W' + p(t)\, W = 0$$

is a 1st order ODE in the Wronskian det as a fnc of $t$.

By Separation of variables:

$$\frac{W'}{W} = -p(t) \Rightarrow \ln|W| = -\int p(t)\,dt + K$$

$$\Rightarrow W = C e^{-\int p(t)\,dt}$$

Either $W = 0$ on all of $I$ or $W \neq 0$ on all of $I$.