~~Wednes~~ 110.302 Lecture 10: 9/23/15

I will relegate the discussion of Section 2.8 to a worksheet posted. The theory can be deep (though very interesting).

But the main takeaway is its usefulness in

① seeing where a 1^{st} order ODE is "nice"

② Understanding the intricacies of the theory even at this early stage.

<u>Section 3.1</u> ~~The~~ A general form for a 2^{nd} order ODE is

$$(*) \qquad y'' = f(t, y, y')$$

for some function f.

<u>Def</u> A 2^{nd} order ODE is called <u>linear</u> if it can be written

$$y'' + p(t)y' + q(t)y = g(t)$$

(so that $f(t,y,y') = g(t) - p(t)y' - q(t)$ in $(*)$)

$(**)$ (actually $P(t)y'' + Q(t)y' + R(t)y = G(t) \ldots$)

<u>Notes</u> ① f needs to be linear in both y and y'.

② If ODE is not linear, it is called non-linear.

<u>Def</u> If $g(t) \equiv 0$, then a linear ODE is called <u>homogeneous</u>.

<u>Def</u>. An IVP with a 2^{nd} order ODE contains 2 pieces of initial data, usually

$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Q: Why?

In general, it is hard or impossible to solve a 2^{nd} order ODE

Even linear is very difficult in general!

One type that is solvable: Constant coefficients

Let (**) have $P(t) \equiv a$, $Q(t) \equiv b$, $R(t) \equiv c$, and suppose $G(t) \equiv 0$ (homogeneous).

Then ODE is $\quad ay'' + by' + cy = 0 \quad$ (A)

Q: First think, what kinds of functions would possibly be solutions to this kind of ODE?

- Polynomials? Power Functions?
- Trig functions?
- Exponentials?
- Logarithms?

ex Suppose $a=1$, $b=0$, $c=-1$. Then (**) is $y''-y=0$, or $y''=y$.

Solutions? Here $y(t)=e^t$ and $y(t)=e^{-t}$ both solve $y''-y=0$.

How about e^t+e^{-t}? $2e^t-3e^{-t}$?

Here $y(t)=c_1e^t+c_2e^{-t}$ is a solution for any choice of $c_1, c_2 \in \mathbb{R}$.

What if IVP was $y''-y=0$, $y(0)=3$, $y'(0)=4$?

$$\underbrace{\begin{aligned} &\text{Then} \quad y(0)=3=c_1e^0+c_2e^{-0}=c_1+c_2 \\ &\text{and} \quad y'(0)=4=c_1e^0-c_2e^{-0}=c_1-c_2 \end{aligned}}$$

$$\left.\begin{aligned} 3&=c_1+c_2 \\ 4&=c_1-c_2 \end{aligned}\right\} \quad \begin{aligned} c_1&=\tfrac{7}{2} \\ c_2&=\tfrac{-1}{2} \end{aligned}$$

And the particular solution to IVP is

$$y(t)=\tfrac{7}{2}e^t-\tfrac{1}{2}e^{-t}.$$

ex 2 $2y'' + 8y' - 10y = 0$

Here $y(t) = e^{t}$ and $y(t) = e^{-5t}$ both work! Check this....

AND so does $y(t) = c_1 e^{t} + c_2 e^{-5t}$.

Is there a pattern?

For $ay'' + by' + cy = 0$, assume the solution is exponential (is this a good idea?) and looks like $y(t) = e^{rt}$ where r is an unknown parameter.

Then substituting $y(t)$ and its derivatives into the ODE, we get

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

(A) or $ar^2 + br + c = 0$. Any valid values for r must satisfy $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Recognize this?

For a 2nd-order homogeneous ODE with constant coefficients, (A) is called the <u>characteristic</u> <u>equation</u>

<u>Important</u>: When the two roots r_1, r_2 of the char. eqn are <u>real</u> and <u>distinct</u>, then the general solution to the ODE is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

(we will need to make sure this is the case later)

<u>ex1</u>. $y'' - y = 0$. Here $a=1, b=0, c=1$, and characteristic eqn is $r^2 - 1 = 0$, with roots $r = -1, 1$. Gen soln is

$$y(t) = c_1 e^{t} + c_2 e^{-t}$$

<u>ex2</u> Char eqn is $2r^2 + 8r - 10 = 0$ which factors to $(2r-2)(r+5) = 0$, so $r = 1, -5$

$$y(t) = c_1 e^{t} + c_2 e^{-5t}.$$