~~Re~~ 110.302 Lecture 9: 9/21/15 III

Suppose a 1^{st} order ODE has the form

$$M(x,y) + N(x,y)\,y' = 0 \qquad (*)$$

Then ($\star$) and ($*$) are the same under the condition that there exists a function $\varphi(x,y)$ where

$$① \ \frac{\partial\varphi}{\partial x}(x,y) = M(x,y)$$

$$② \ \frac{\partial\varphi}{\partial y}(x,y) = N(x,y)$$

If true, then

$$\begin{array}{ccccccc} M(x,y) & + & N(x,y) & y' & = 0 & & (*) \\ \shortparallel & & \shortparallel & \shortparallel & & & \\ \frac{\partial\varphi}{\partial x}(x,y) & + & \frac{\partial\varphi}{\partial y}(x,y) & \frac{dy}{dx} & = 0 & & (\star) \end{array}$$

If we can equate the two, then the general implicit solution to

$$M(x,y) + N(x,y)\,y' = 0$$

would be $\varphi(x,y) = c$

(since then $\displaystyle \frac{d\varphi}{dx}(x,y) = \underbrace{\underbrace{\frac{\partial\varphi}{\partial x}(x,y)}_{M} + \underbrace{\frac{\partial\varphi}{\partial y}(x,y)}_{N}\frac{dy}{dx} = 0}_{\text{original ODE!}}$

Ex 1 in book

Given the ODE $2x + y^2 + 2xyy' = 0$, note that in the form (*), $M(x,y) = 2x + y^2$
$N(x,y) = 2xy$.

And since $\exists\ \psi(x,y) = x^2 + xy^2$, where

$$\frac{\partial \psi}{\partial x}(x,y) = \underbrace{2x + y^2}_{M} \qquad \frac{\partial \psi}{\partial y}(x,y) = \underbrace{2xy}_{N}$$

the original ODE can be written

$$\frac{d\psi}{dx}(x,y) = 0 = \frac{d}{dx}\left(x^2 + xy^2\right)$$

Integrate this wrt x and get

$$\psi(x,y) = x^2 + xy^2 = c$$

as our general (implicit) solution.

Q: How do we know when such a $\psi(x,y)$ exists?

Q: How to find it when we do know?

<u>Calculus III Thm</u> Let $\psi(x,y)$ have continuous partial derivatives in some open region. Then

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)$$

i.e., the mixed 2^{nd} partials are equal.

~~Put all of this together and we get the following~~

Now if we had the ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

and knew there was a function $\psi(x,y)$ where $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$, then the last criterion is

$$\frac{\partial}{\partial x}(N) = \frac{\partial}{\partial y}(M), \text{ or } \boxed{N_x = M_y}$$

<u>Def</u> The ODE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is called <u>exact</u> on a region /

$$R = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, \begin{matrix} \alpha < x < \beta \\ \gamma < y < \delta \end{matrix} \right\}.$$

If ① M, N, M_y, N_x are C^0 on R, and

② $M_y = N_x$ on R.

<u>Thm</u> Let $M(x,y) + N(x,y)y' = 0$ be exact on some open region R. Then $\exists$ a function $\psi(x,y)$ which is diff on R where

① $\frac{\partial \psi}{\partial x} = M$ ② $\frac{\partial \psi}{\partial y} = N$ and

$\psi(x,y) = C$ is the general implicit solution to the ODE on R.

In practice?

ex. Solve $(3x^2-2xy+2)+(6y^2-x^2+3)y'=0$

$$y(1)=0$$

Strategy First, we verify exactness. Then we integrate to find the function whose level sets comprise solutions to the ODE.

Solution Here $M(x,y)=3x^2-2xy+2$
$N(x,y)=6y^2-x^2+3$

and since $M_y=\frac{\partial M}{\partial y}=-2x=\frac{\partial N}{\partial x}=N_x$

the ODE is exact.

And since M, N, M_y, N_x are all C^0 on $\mathbb{R}^2$, by Thm $\varphi(x,y)$ will exist near $(1,0)\in\mathbb{R}^2$

To find $\varphi(x,y)$, note that $\frac{\partial\varphi}{\partial x}=M$, $\frac{\partial\varphi}{\partial y}=N$

Integrate M wrt x:

$$\int M\,dx=\int\frac{\partial\varphi}{\partial x}dx=\int(3x^2-2xy+2)\,dx=x^3-x^2y+2x+h(y)$$

why? ⟶

Hence $\varphi(x,y) = x^3 - x^2 y + 2x + h(y)$
for some unknown function $h(y)$.

To find $h(y)$, note that $\frac{\partial \varphi}{\partial y} = N$:

$$\frac{\partial \varphi}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(x^3 - x^2 y + 2x + h(y)\right)$$

$$= -x^2 + h'(y) =$$

$$= N(x,y) = 6y^2 - x^2 + 3$$

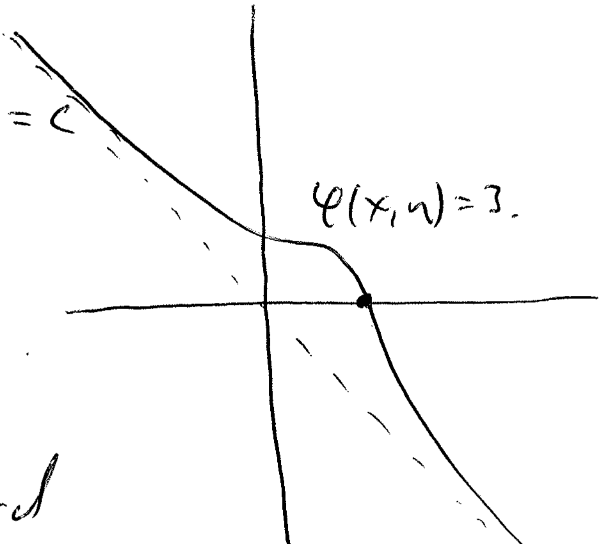
Hence $h'(y) = 6y^2 + 3$, or $h(y) = 2y^3 + 3y + \text{const.}$

Thus our general implicit solution is

$$\varphi(x,y) = x^3 - x^2 y + 2x + 3y + 2y^3 = C$$

Our particular solution
is $\varphi(1,0) = 1 - 0 + 2 + 0 + 0 = C$
or $C = 3$, so

$$\varphi(x,y) = 3 = x^3 - x^2 y + 2x + 3y + 2y^3$$



Determining a valid interval
is difficult here.

In practice?

ex. Solve $2x+y^2+2xyy'=0$, $y(1)=1$.

Strategy: First, we verify exactness. Then we integrate to find the function whose level sets comprise solutions to the ODE.

Solution: Here $M(x,y)=2x+y^2$

$N(x,y)=2xy$.

and since $M_y = \frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x} = N_x$

the ODE is exact.

And since M, N, M_y, N_x are all C^0 on $\mathbb{R}^2$,

by Thm, ~~solutions~~ $\varphi(x,y)$ will exist

near $(1,1) \in \mathbb{R}^2$.

To find $\varphi(x,y)$, note that $\frac{\partial \varphi}{\partial x} = M$, $\frac{\partial \varphi}{\partial y} = N$

Integrate M with respect to x:

$$\int M\,dx = \int \frac{\partial \varphi}{\partial x}\,dx = \int (2x+y^2)\,dx$$

$$= x^2 + xy^2 + h(y)$$ why?

Hence $\varphi(x,y) = x^2 + xy^2 + h(y)$ for some unknown function $h(y)$.

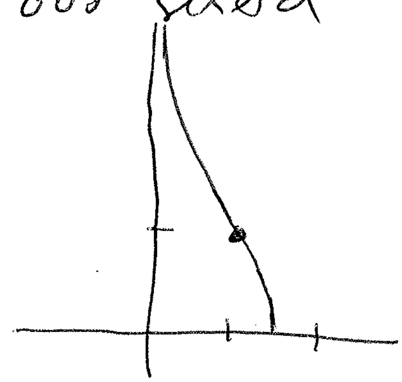
To find $h(y)$, note also that $N = \frac{\partial \varphi}{\partial y}$:

$$\frac{\partial \varphi}{\partial y}(x,y) = \frac{d}{dy}\left(x^2 + xy^2 + h(y)\right)$$
$$= 2xy + h'(y)$$
$$= N(x,y) = 2xy.$$

Hence $h'(y) = 0$, or $h(y) = \text{constant}$.

Thus $\varphi(x,y) = x^2 + xy^2 + \text{constant}$, and

$\varphi(x,y) = x^2 + xy^2 = c$ is our general implicit solution.



The particular solution solves

$$\varphi(1,1) = 1^2 + (1)(1)^2 = 2.$$

The solution to $2x + y^2 + 2xyy' = 0$, $y(1) = 1$

$x^2 + xy^2 = 2$, valid on $x \in (0, \sqrt{2})$

Check this!

Notes ① Sometimes, the ODE is in differential form:

ex. $(ye^{2xy}+x)\,dx + xe^{2xy}\,dy = 0$

is exact, since $M(x,y) = ye^{2xy}+x$

$N(x,y) = xe^{2xy}$

and
$$\left.\begin{aligned} M_y &= e^{2xy} + 2xye^{2xy} \\ N_x &= e^{2xy} + 2xye^{2xy} \end{aligned}\right\} \text{equal.}$$

② Caution: Sometimes a non exact 1st order ODE can be made exact via an integration factor:

ex. $dx + \left(\frac{x}{y} - \sin y\right)dy = 0$ is not exact

$M_y = 0 \neq \frac{1}{y} = N_x.$

But $y\left[dx + \left(\frac{x}{y} - \sin y\right)dy = 0\right]$

$y\,dx + (x - y\sin y)\,dy = 0$

is exact since now

$M_y = 1 = \frac{d}{dx}[x - y\sin y] = 1 = N_x.$

We won't focus on this last technique.