110.302 Lecture 8: 9/18/15 I

<u>Bifurcations</u>

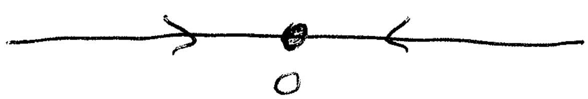
Consider the autonomous $y' = f(a,y)$ where "a" is a parameter (an unknown constant).

The number and classification of equilibria may depend on the value of "a"

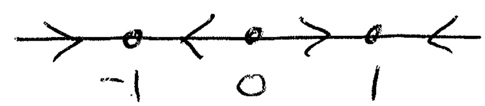
ex. $\dot{y} = ay - y^3 = y(a - y^2)$

Here, for $a<0$ and $a>0$ the number of equilibria are different (also the type)

$a = -1 < 0$



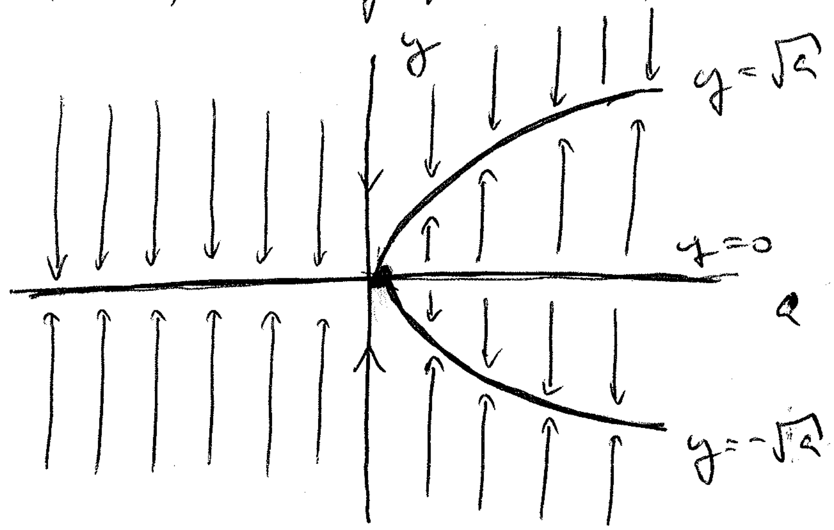
$a = 1 > 0$



Here $f(-1,y) = -y(1+y^2)$ and $y(t) \equiv 0$ is the only equilibrium. It is asympt. stable

Here $f(1,y) = y(1-y^2)$
$= y(1-y)(1+y)$
and equilibria exist at $y = -1, 0, 1$. $y(t) \equiv 0$ is unstable here.

We can study how the parameter affects equilibria via a <u>bifurcation</u> <u>diagram</u>: a graph of equilibria in relation to parameter value in the ay-plane for $y' = f(a,y)$.

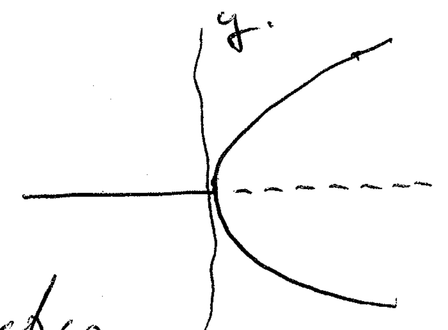


<u>Properties</u>

- each vertical slice here <u>is</u> the phase line of $y' = f(a,y)$ for that value of "a".
- As a varies, equilibria trace out curves of fixed pts. Found by solving $f(a,y)=0$.
- Special values of "a" where the number of equilibria and/or the stability change are called <u>bifurcation</u> <u>values</u> of "a".
- Here, curves for $a>0$ correspond to solutions to $f(a,y) = y(a-y^2)=0$, or to $\boxed{y=0}$ and $a=y^2$ or $\boxed{y=\pm\sqrt{a}}$

- Here, the only bifurcation value of a is $a=0$.
- Here we use solid lines for all equilibria, and vertical arrows to denote stability.

Book uses solid for asympt. stable curves and dotted for unstable curves.



- This kind of bifurcation is called a <u>pitchfork bifurcation</u>

<u>why?</u>

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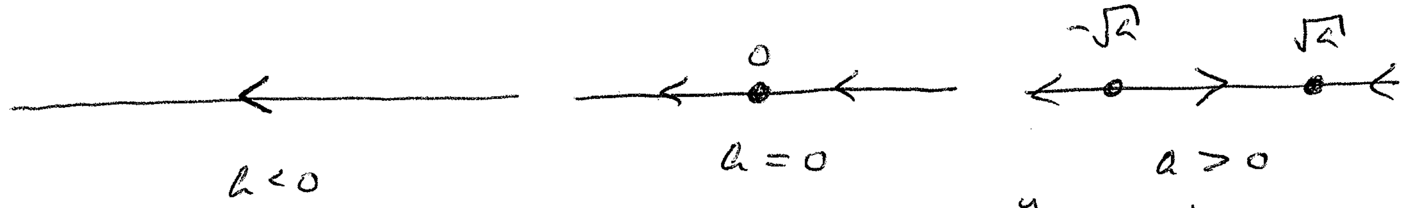
ex. $\dot{y} = a - y^2$



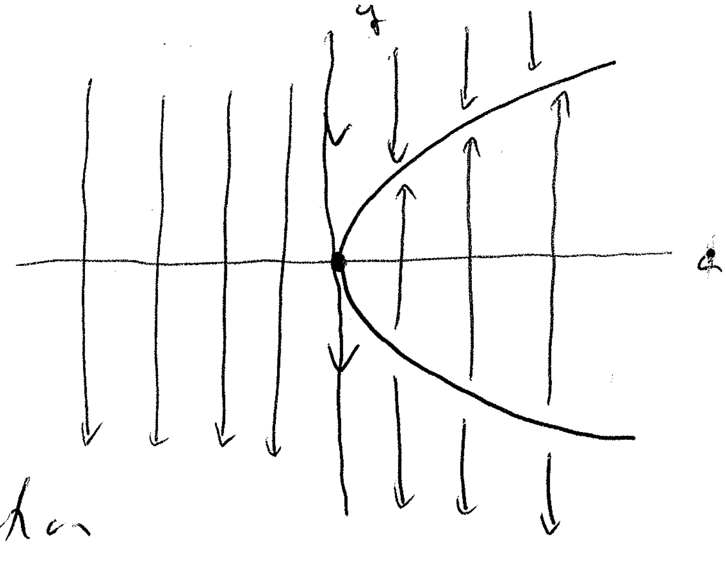
Lines of equilibria

solve $a = y^2$ or

$y = \sqrt{a}$, $y = -\sqrt{a}$

only for $a \geq 0$

Called a <u>saddle-node</u> or <u>creation</u> bifurcation



ex. Basic model of a Laser (simple)

$$\dot{n} = (GN_0 - k)n - Gn^2$$

models a basic simple laser, where

$n(t)$ = # of photons at time t

N_0, k, G are positive constants

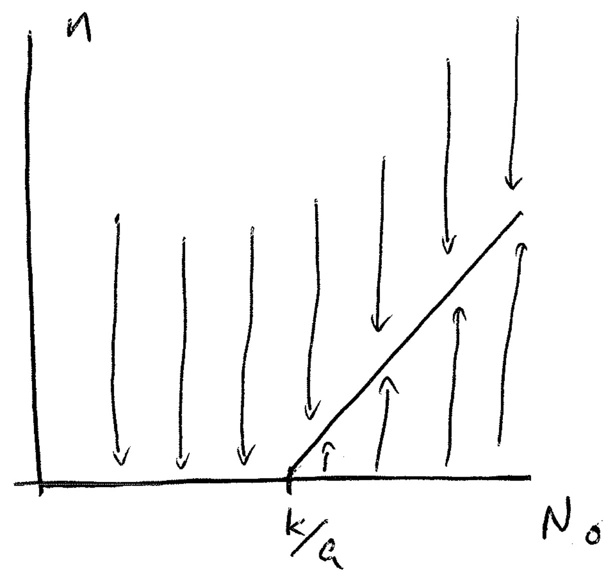
We study how the eqn is affected by $N_0 \geq 0$

Here equilibria are at

$$n(GN_0 - k - Gn) = 0$$

namely $n = 0$

$n = N_0 - k/G$



(I) when $N_0 < k/G$,

$(GN_0 - k) < 0$,

so $\dot{n} < 0$.

$n(t) \equiv 0$ is a sink.

(II) when $N_0 > k/G$, $GN_0 - k > 0 \Rightarrow$ for small n,

$GN_0 - k - Gn > 0$, or $N_0 - \frac{k}{G} - n > 0$, so

for small n, $\dot{n} > 0$. <u>etc.</u>

Change tack

Recall for any ~~eq~~ equation involving x, y,

- we can bring all terms to one side of the equation and create an equivalent equation

$$\varphi(x,y) = 0$$

for $\varphi(x,y)$ a function of 2 variables. Then the curve in the xy-plane satisfying this equation is called the 0-level set of φ

ex. $y^2 = 1 - x^2$. We view this eqn as the 0-level set of the function $\varphi(x,y) = x^2 + y^2 - 1 = 0$

- We can view y as an implicit function of x.

In either case, the ~~set~~ graph of the original equation (or the $\varphi(x,y) = 0$) is a curve in xy-plane that in general will not look like a function.

We can calculate the tangent lines to this graph via differentiation in either interpretation:

ex. $x^2 + xy^2 = 4$, or $\varphi(x,y) = 0$, $\varphi(x,y) = x^2 + xy^2 - 4$

Implicit diff $\frac{d}{dx}(x^2 + xy^2 = 4) \Rightarrow \underbrace{2x + y^2 + 2xy\frac{dy}{dx} = 0}_{(\bigstar)}$

Calc III $\frac{d\varphi}{dx}(x,y) = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\cdot\frac{dy}{dx} = \underbrace{\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\cdot y'}_{(\bigstar)}$

when we think of y as an implicit fnc of x.

Suppose a first-order ODE is of the form

$$M(x,y) + N(x,y)\,y' = 0 \qquad (*)$$

Then ($\bigstar$) and ($*$) are the same under the condition that there exists a function $\varphi(x,y)$, where ① $\frac{\partial\varphi}{\partial x}(x,y) = M(x,y)$

② $\frac{\partial\varphi}{\partial y}(x,y) = N(x,y)$

So that $M(x,y) + N(x,y)\,y' = 0 = \frac{d\varphi}{dx} = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\,y'$

If this is the case, then the ODE $(*)$ can be rewritten as $\frac{d\psi}{dx} = 0$, or

$\psi(x,y) = C$, a constant.

Thus solving the ODE at least implicitly.

Ex 1 (cont).

Notice that $2x + y^2 + 2xyy' = 0$ is of the form $M(x,y) + N(x,y)y' = 0$ with

$$M(x,y) = 2x + y^2$$
$$N(x,y) = 2xy.$$

But we also can see that the function $\psi(x,y) = x^2 + xy^2$ has the partials

$$\frac{\partial\psi}{\partial x}(x,y) = 2x + y^2 \quad \frac{\partial\psi}{\partial y}(x,y) = 2xy.$$

Hence $2x + y^2 + 2xyy' = 0$ can be written

$$\frac{d\psi}{dx} = 0 = \frac{d}{dx}(x^2 + 2xy^2)$$

if we assume that y is an implicit function of x.

Here we can (assuming y is an implicit function of x) integrate $\frac{d\psi}{dx}(x,y) = 0$ wrt x to get

$$\int \frac{d\psi}{dx}(x, y(x))\, dx = \int 0\, dx$$

$$\psi(x,y) = x^2 + xy^2 = C$$

This is the general implicit solution to

$$2x + y^2 + 2xyy' = 0$$

Q: How do we know such a $\psi(x,y)$ may exist and if so how to find it?

Calc III Thm Let $\psi(x,y)$ have continuous partial derivatives in some open region. Then

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)$$

i.e. Mixed partials are equal.