(3) Equilibrium Solutions

At any place y_0 where $f(y_0) = 0$, then $y'(t) = 0$ here, and this $y(t) = y_0$ is a constant solution (or equilibrium, or steady-state solution).

Its graph is a horizontal line and is an isocline.

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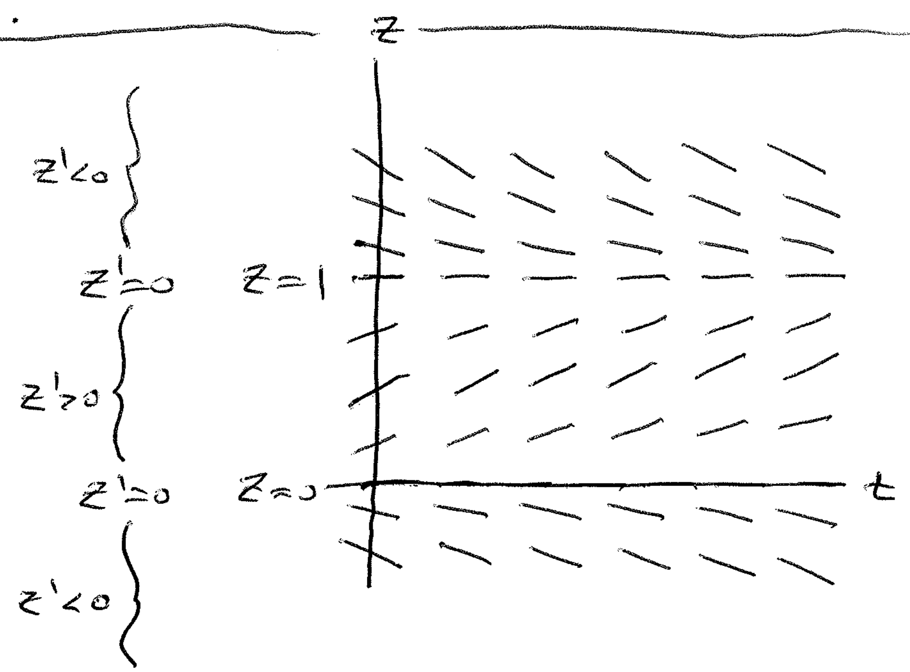
ex $\quad z' = z(1-z)$

Here $z(t) = 0$ and $z(t) = 1$ are both equilibrium solutions



And in between the equilibria, the sign of z' does not change. Hence solutions

(A) are trapped between equilibria, and

(B) always travel in the same direction.

In the example, we can say the following without solving:

(I) Solutions exist and are unique everywhere
($f(z)$ and $f'(z)$ are polynomials)

(II) Equilibria only at $z=0$ and $z=1$.

(III) Any solution that passes through $0<z_0<1$ will tend toward the equilibria $z(t)=1$.

Any solution that starts at $z_0<0$ will tend to $-\infty$

Any solution that starts at $z_0>1$ will tend to $z(t)=1$

Here we can say
$$\lim_{t\to\infty} z(t) = \begin{cases} 1 & z_0>0 \\ 0 & z_0=0 \\ -\infty & z_0<0 \end{cases}$$

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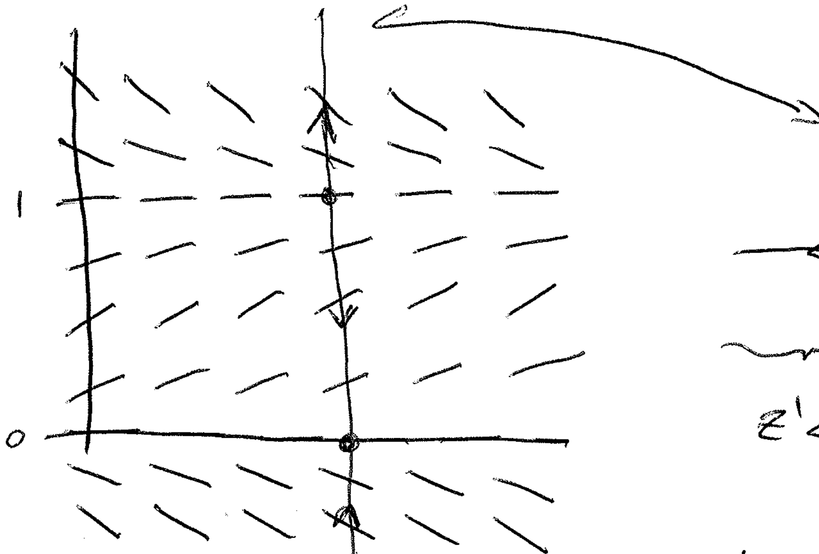
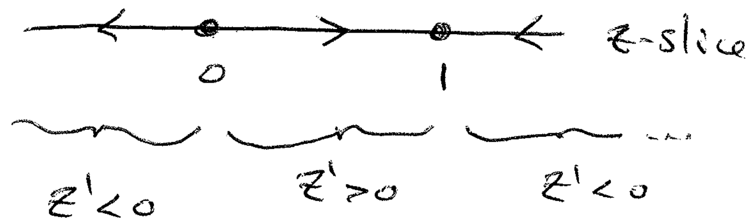
(4) <u>Phase line</u>

Any vertical slice through the slope field gives you all information about long term behavior of solutions:



z-slice

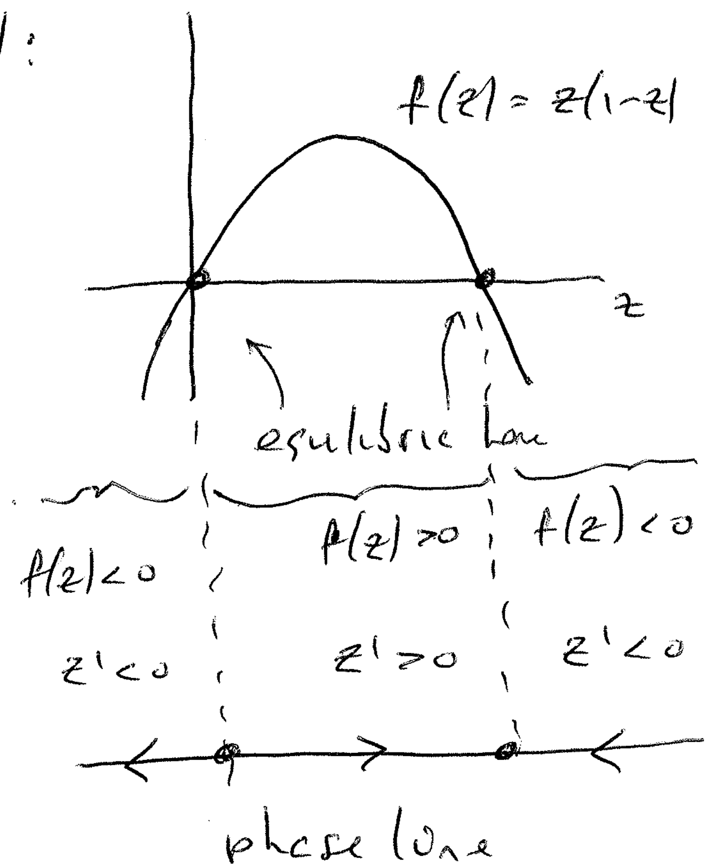
Phase line of $z'=z(1-z)$



Here, the phase line is a schematic that determines all long term behavior of the autonomous $z'=f(z)$.

without a slope field, still easy to see phase line: Graph $f(z)$:

$$z' = z(1-z) = f(z)$$



Def. For $y'=f(y)$, the set $\{y \in \mathbb{R} \mid f(y)=0\}$ is the set of <u>critical pts</u> for the ODE. (equilibrium solutions)

Critical pts (equilibrium solutions) can be classified by how solutions behave around them:

Let y_* be a critical pt for $y' = f(y)$, and let $N_\varepsilon(y_*) = \{y \in \mathbb{R} \mid |y - y_*| < \varepsilon\}$ be an ε-neighborhood of y_*, ~~[illegible]~~

(a) If there is an $\varepsilon > 0$ where for all $y \in N_\varepsilon(y_*)$

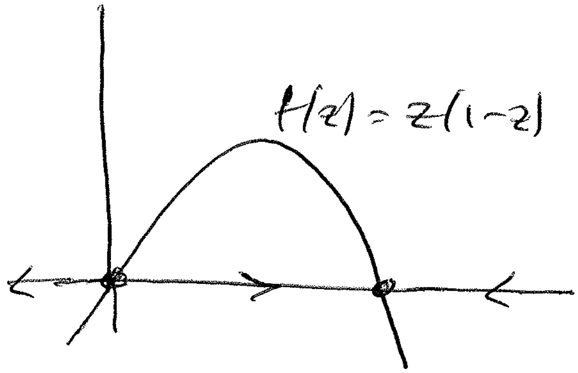
$\lim_{t \to \infty} y(t) = y_* \implies y_*$ is <u>asymptotically stable</u>

(b) If there is an $\varepsilon > 0$ where for all $y \in N_\varepsilon(y_*)$

$\lim_{t \to -\infty} y(t) = y_* \implies y_*$ is unstable

(c) If asympt. stable on one side and unstable on the other, then y_* is semi-stable.

ex. $z' = z(1-z)$. Here critical pts are $z = 0, 1$. And



Here $z(t) = 1$ is asymptotically stable and $z(t) = 0$ is unstable.

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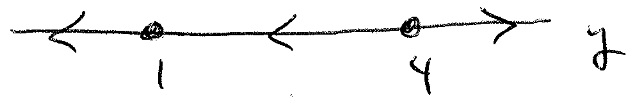
ex. $y' = (1-y)^2 (y-4)$

Here, critical pts at $y = 1, 4$.

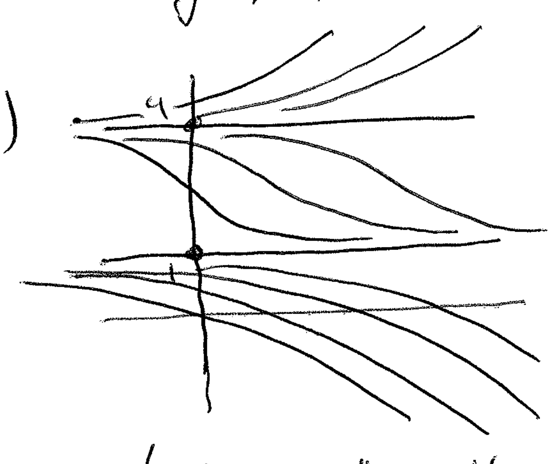
Phase line is



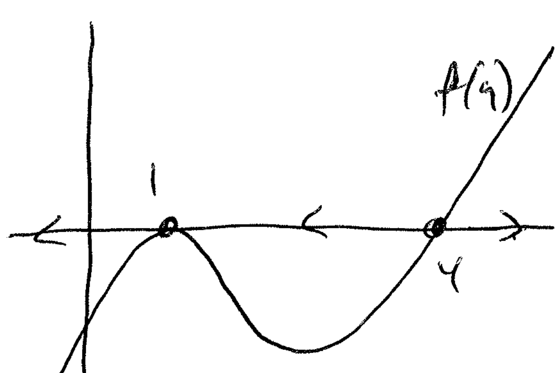
(check a pt in each interval formed by critical pts.

$y(t) = 4$ is unstable

$y(t) = 1$ is semistable.



Graph of ~~g~~ $f(y) = (1-y)^2 (y-4)$



$$\lim_{t \to \infty} y(t) = \begin{cases} \infty & y_0 > 4 \\ 4 & y_0 = 4 \\ 1 & 1 \le y_0 < 4 \\ -\infty & y_0 < 1 \end{cases}$$

When graphing the $f(y)$ in $y' = f(y)$ and in constructing phase lines, some pattern develops:

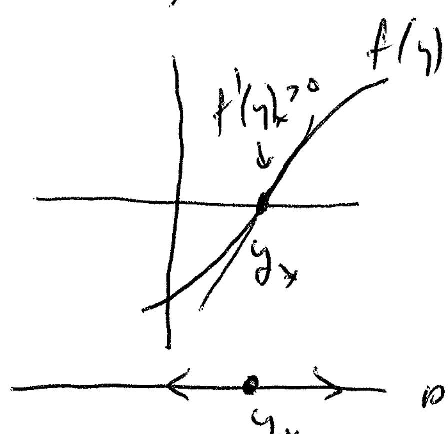
Let y_* be an equilibrium for $y' = f(y)$ (thus $f(y_*) = 0$).

For y "near" y_0, $y' = f(y) \cong \underbrace{\cancel{f(y_*)}^{\,0} + f'(y_*)(y - y_*)}_{1^{\text{st}} \text{ Taylor approx to } f(y) \text{ at } y_*}$

Case: Suppose $f'(y_*) > 0$

$\Longrightarrow$ for $y > y_*$, $y' > 0$

$y < y_*$, $y' < 0$



phase line

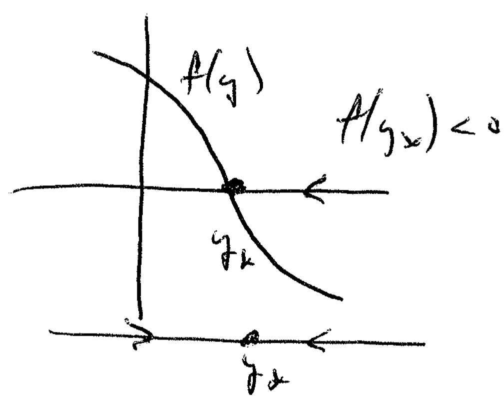
All nearby solutions move away

$\Longrightarrow$ y_* is an unstable node or source

Case 2: for $f'(y_*) < 0$

$\Longrightarrow$ for $y > y_*$ $y' < 0$

$y < y_*$ $y' > 0$

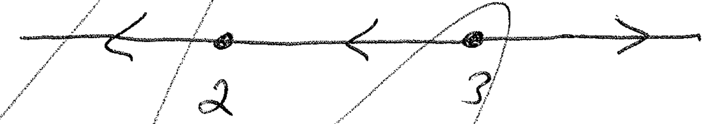


All nearby sol'ns converge to y_*.

$\Longrightarrow$ asymptotically stable or <u>sink</u>.

Case 3: $f'(y_*) = 0$. Need more information.

ex. $y' = (y-2)^2(y-3)$



Here $y=3$ is a repeller

$y=2$ is a semi-stable node.

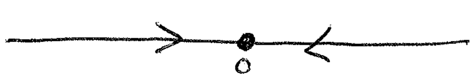
Bifurcations

Consider the autonomous $y' = f(a,y)$ where "a" is a parameter. (an unknown constant)

Equilibria may depend on the value of a. (~~both~~ in location, # and stability type)

ex $\dot{y} = ay - y^3 = y(a-y^2)$

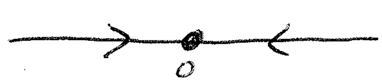
$a<0$



ex. $a=-1$ $\dot{y} = -y(1+y^2)$

equil. $y(t) \equiv 0$ is asympt stable

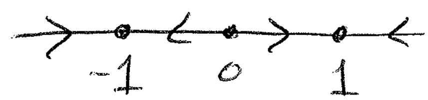
$a=0$



$\dot{y} = -y^3$

$y(t) \equiv 0$ sink

$a>0$

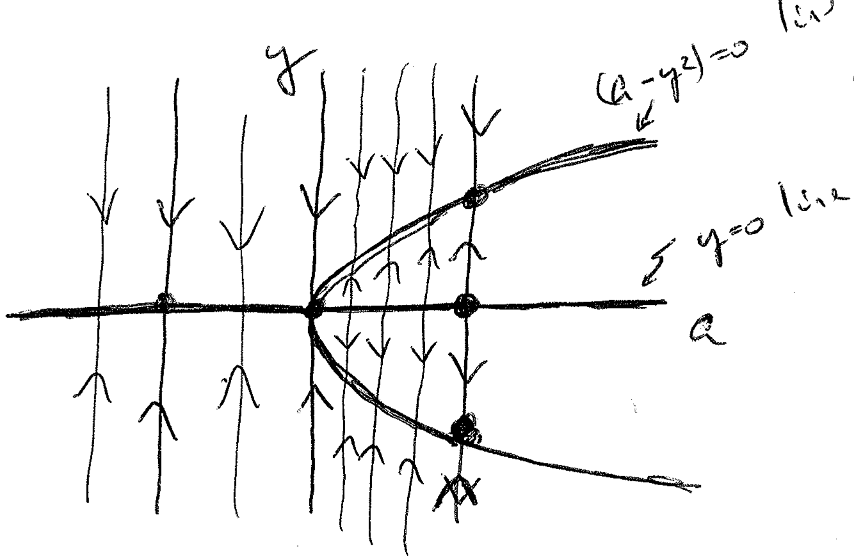


ex: $a=1$ $\dot{y} = y(1-y)(1+y)$

$y(t) \equiv 0$ source

$y(t) = \pm 1$ sink.

We can study how "a" affects equilibria via a "bifurcation diagram": a graph of equilibria (and stability) in relation to parameter value ?



Properties

- each vertical slice is the phase line for a value of "a".
- As a varies, equilibria trace out ~~lines~~ curves of fixed pts.
- values of a in which the stability and/or # of equilibria change are called <u>bifurcation</u> <u>values</u> of a.
- $\boxed{\text{These are rare!}}$ stability cannot change outside of these.

- ~~lines~~ curves of fixed pts can be found by solving $f(a,y)=0$ locally:

$$f(a,y) = y(a-y^2)$$
$$=0$$

when $y=0$ or $y^2=a$

(a-axis) (sideways parabola)

$y=\sqrt{a}$

$y=-\sqrt{a}$

Here the only ~~point is~~ bif value is $a=0$.

Note: In book, stable lines are solid, unstable are dotted.
Here, ~~stable lines~~ stability is marked by phase line arrows.

Notes ① In book, stable lines are solid, unstable are dashed. Here, I use phase lines to denote stability.

② Stability cannot change outside of bif. pts, and cannot change far away

③ $a=0$ is called a pitchfork bifur. for $\dot{y} = ay - y^3$.

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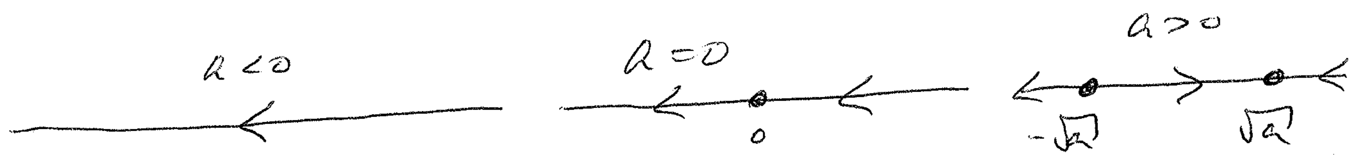
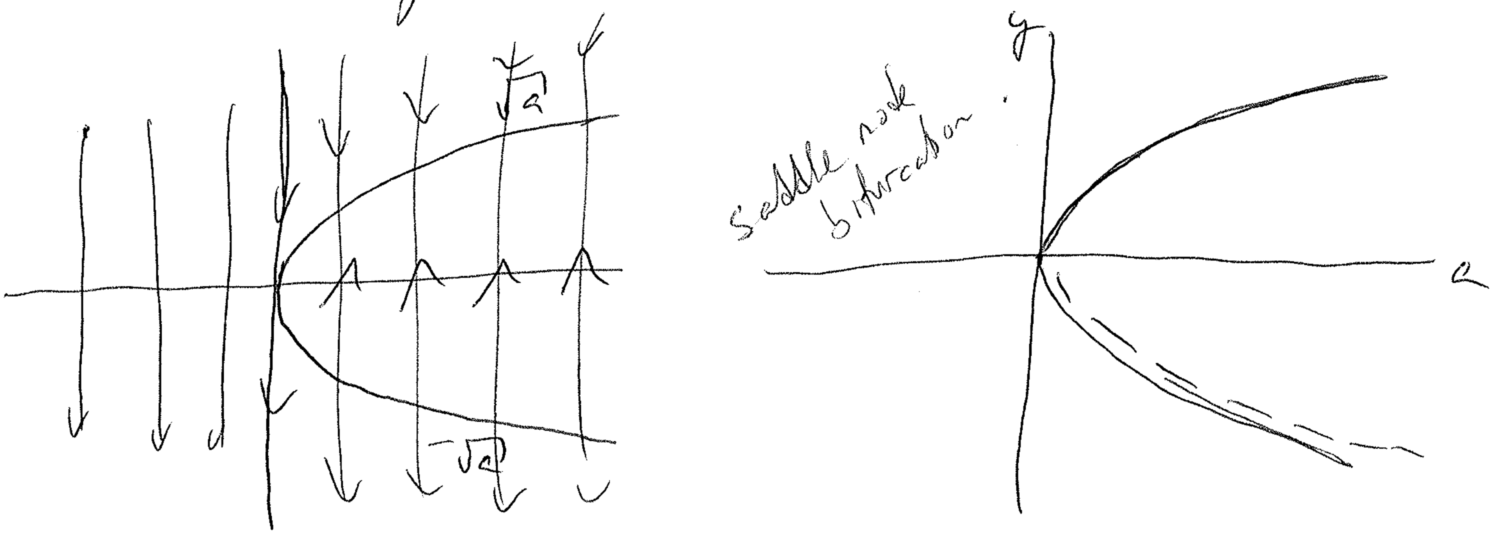
ex. $\dot{y} = a - y^2$



lines of ~~stabilit~~ equil can only be

$a - y^2 = 0 \Rightarrow a = y^2$ (sideways par)



~~Note~~: Basic physics behind a Laser

example $\dot{n} = (GN_0 - k)n - Gn^2$

is an eqn involving laser physics

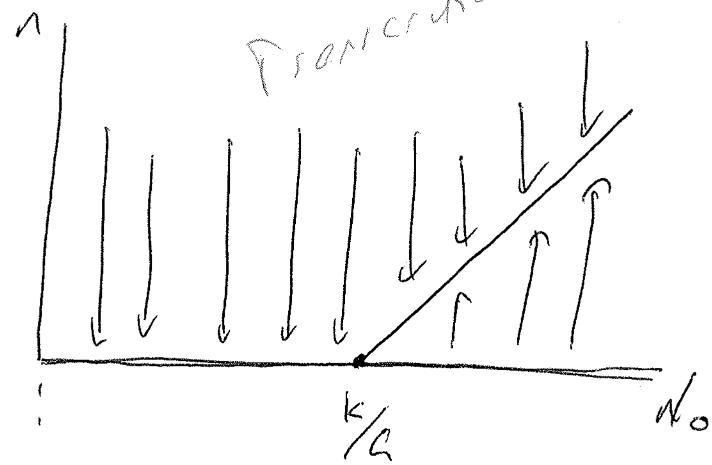
where G, N_0, k are pos constants. ~~$G, k > 0$~~ (N_0 is variable)

and $n(t)$ is the # of photons (always $n \geq 0$).

equil solns are $n(GN_0 - k - Gn) = 0$

$$n = 0, \quad n = N_0 - k/G.$$

Bifurcation diag is



I when $N_0 < k/G$

$\Rightarrow GN_0 - k < 0$

$\Rightarrow \dot{n} < 0$

$\Rightarrow n = 0$ is sink

II when $N_0 > k/G$, $GN_0 - k > 0$

$\Rightarrow GN_0 - k - Gn$

$N_0 - \frac{k}{G} - n > 0$ for small n.

so $GN_0 - k - Gn > 0 \Rightarrow \dot{n} > 0$ for small n.

$\Rightarrow n = 0$ is a source

if $N_0 - k/G - n < 0$ for $n > N_0 - k/G$.

$\Rightarrow$ line $N_0 - k/G$ is a sink