New Structure type: Autonomous

Suppose in $y' = f(t,y)$, f is only a function of y: $y' = f(y)$

Such an ODE IS separable, but $\frac{1}{f(y)}\frac{dy}{dt} = 1$ may still be hard to integrate.

An ODE of the form $y' = f(y)$ is called autonomous: t is not explicitly present in the equation.

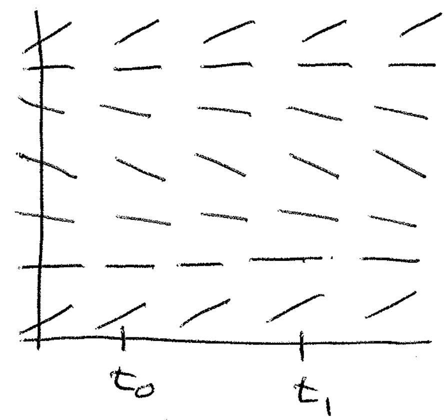
exs $\dot{x} = x^{4/3}$, $y' = ky$, k a constant

$\frac{dz}{dt} = z(1-z)$ (Logistic eqn).

Here, even without solving, properties of autonomous ODEs allow for effective study.

Properties of autonomous $y' = f(y)$

① <u>Structure of slope field.</u>



- Slope field doesn't change in t-direction. (different t's have same look)
  - Every vertical slice <u>looks the same</u>.
- Every horizontal slice is an <u>isocline</u>: a curve along which all slopes of solution curves are the same.

② <u>Existence and uniqueness:</u>

Since here f is only a function of y,

- Existence of solutions is assured when $f(y)$ is continuous.
- Uniqueness of solutions is assured when $f'(y) = \frac{df}{dy}$ is continuous.

(Here, $\frac{\partial f}{\partial y}(t,y) = \frac{df}{dy} = f'(y)$ since t does not occur as a variable in f).

Conclusion: No crossings of solutions where $f(y)$ and $f'(y)$ are defined.

(3) Equilibrium Solutions

At any place y_0 where $f(y_0) = 0$, then $y'(t) = 0$ here, and this $y(t) = y_0$ is a constant solution (or equilibrium, or steady-state solution).

Its graph is a horizontal line and is an isocline.

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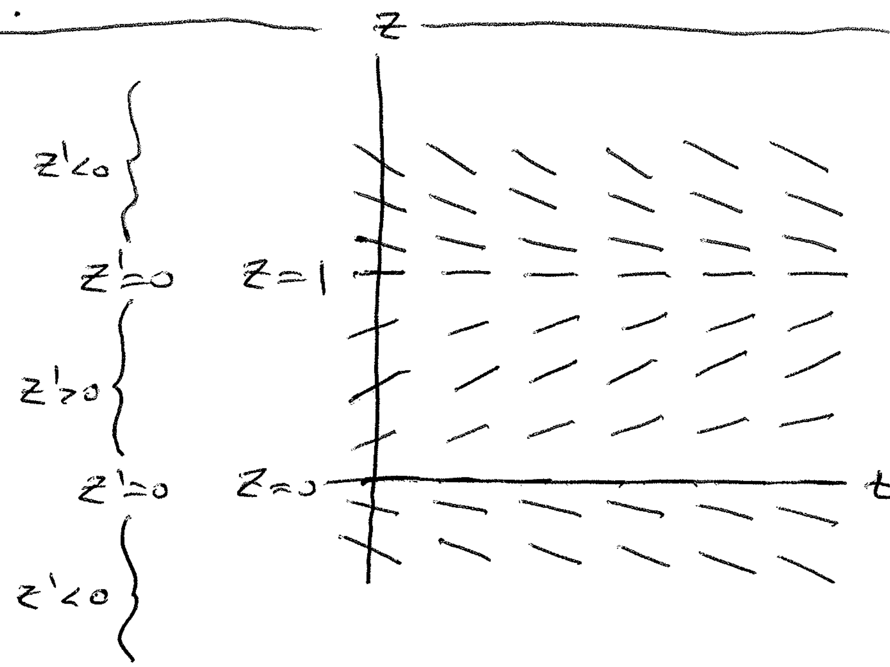
ex $\quad z' = z(1-z)$

Here $z(t) = 0$ and $z(t) = 1$ are both equilibrium solutions



And in between the equilibria, the sign of z' does not change. Hence solutions

(A) are trapped between equilibria, and

(B) always travel in the same direction.

In the example, we can say the following without solving:

(I) Solutions exist and are unique everywhere ($f(z)$ and $f'(z)$ are polynomials)

(II) Equilibria only at $z=0$ and $z=1$.

(III) Any solution that passes through $0<z_0<1$ will tend toward the equilibrium $z(t)=1$.

Any solution that starts at $z_0<0$ will tend to $-\infty$

Any solution that starts at $z_0>1$ will tend to $z(t)=1$

Here we can say $\lim_{t\to\infty} z(t) = \begin{cases} 1 & z_0>0 \\ 0 & z_0=0 \\ -\infty & z_0<0 \end{cases}$

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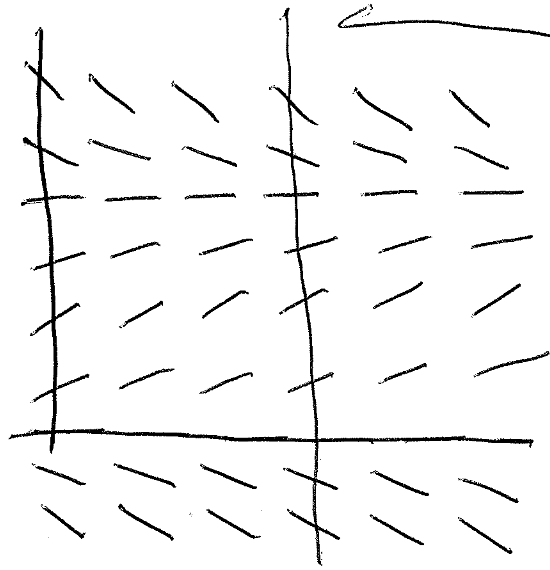
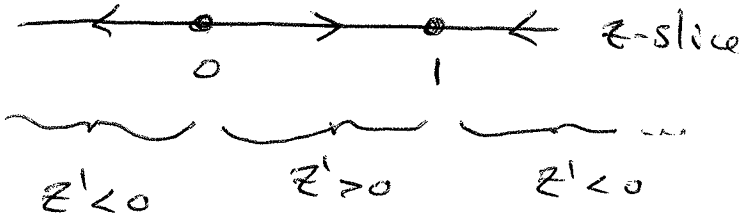
(4) <u>Phase line</u>

Any vertical slice through the slope field gives you all information about long term behavior of solutions:



z-slice

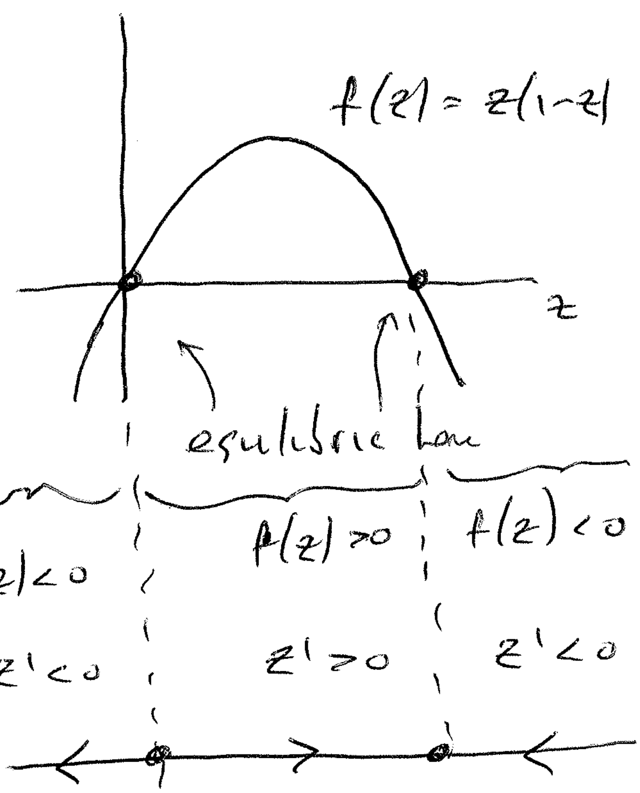
Phase line of $z' = z(1-z)$



Here, the phase line is a schematic that determines all long term behavior of the autonomous $z' = f(z)$.

Without a slope field, still easy to see phase line: Graph $f(z)$:

$$z' = z(1-z) = f(z)$$



phase line

Def. For $y' = f(y)$, the set $\{y \in \mathbb{R} \mid f(y) = 0\}$ is the set of <u>critical pts</u> for the ODE. (equilibrium solutions)

Critical pts (equilibrium solutions) can be classified by how solutions behave around them:

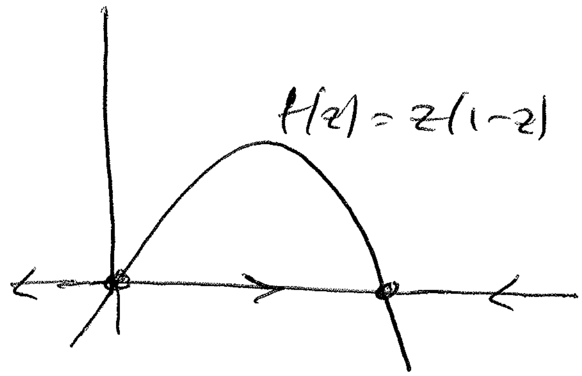
Let y_* be a critical pt for $y' = f(y)$, and let $N_\varepsilon(y_*) = \{ y \in \mathbb{R} \mid |y - y_*| < \varepsilon \}$ be an ε-neighborhood of y_*, ~~[illegible]~~

ⓐ if there is an $\varepsilon > 0$ where for all $y \in N_\varepsilon(y_*)$
$\lim_{t \to \infty} y(t) = y_* \implies y_*$ is <u>asymptotically stable</u>

ⓑ if there is an $\varepsilon > 0$ where for all $y \in N_\varepsilon(y_*)$
$\lim_{t \to -\infty} y(t) = y_* \implies y_*$ is unstable

ⓒ if asympt. stable on one side and unstable on the other, then y_* is semi-stable.

ex. $z' = z(1-z)$. Here critical pts are $z = 0, 1$. And



Here $z(t) = 1$ is asymptotically stable and $z(t) = 0$ is unstable.

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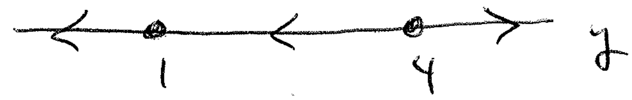
ex. $y' = (1-y)^2(y-4)$

Here, critical pts at $y = 1, 4$.

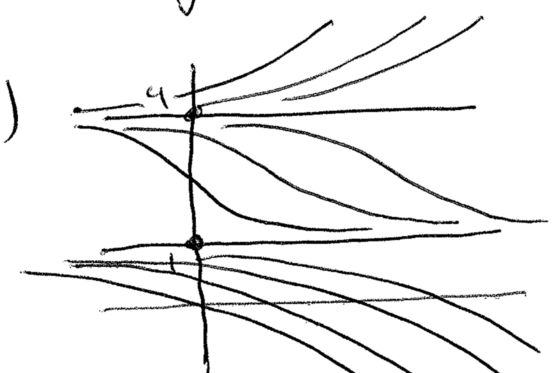
Phase line is

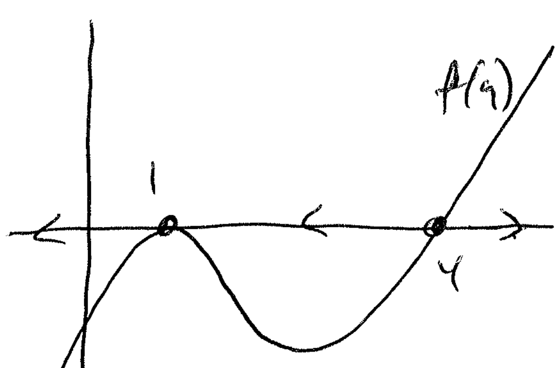


(check a pt in each interval formed by critical pts.)

$y(t) = 4$ is unstable
$y(t) = 1$ is semistable.

Graph of $f(y) = (1-y)^2(y-4)$





$$\lim_{t\to\infty} y(t) = \begin{cases} \infty & y_0 > 4 \\ 4 & y_0 = 4 \\ 1 & 1 \le y_0 < 4 \\ -\infty & y_0 < 1 \end{cases}$$