Lecture 5: Friday, 9/11/15

Section 2.3 deals with applications and the details of how to model a process to construct an ODE.

This is, in general, a difficult process and quite ad hoc. Also it cannot be mastered in such a short time. Read this section.

Section 2.4 deals with issues of non-linearity.

In all examples shown, there has always been a solution. And with initial data, there has always been a unique solution.

Is this always the case??

ex 1. Solve $(y')^2 + 1 = 0$

(Find a function whose derivative squared is -1?)

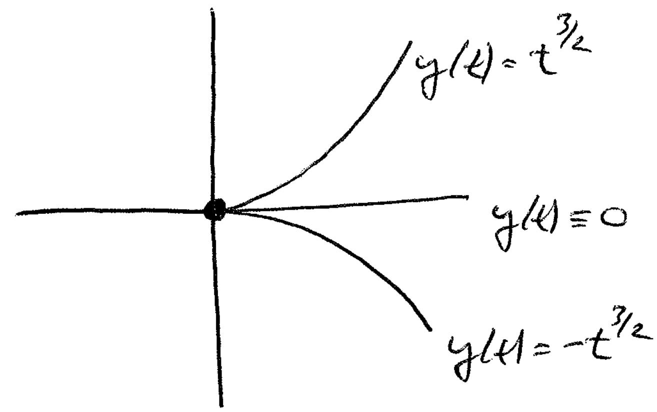
ex 2 Solve $y' = \frac{3}{2} y^{1/3}$, $y(0) = 0$.

Note here that (a) $y(t) \equiv 0$ solves this.

but so does

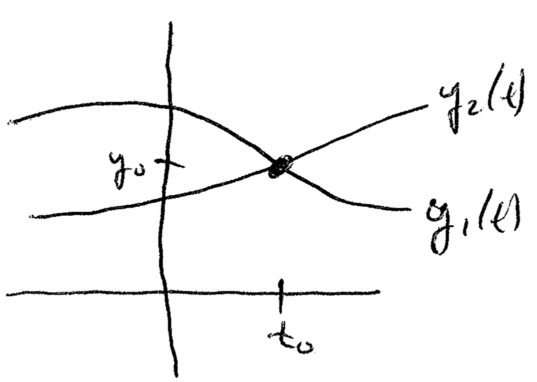
(b) $y(t) = t^{3/2}$

(c) $y(t) = -t^{3/2}$



Since all three of those distinct functions solve the IVP, we say solutions here (at $y(0)=0$) are <u>NOT</u> <u>unique</u>!

Q: If solutions to an ODE are not unique at a pt $y(t_0) = y_0$, then 2 or more solutions that are distinct pass through the pt. What does this say about the predictive power of your model?



There are criteria for ensuring an IVP has solutions (solns exist) and whether they are unique or not (unique is good!).

Let (*) $y'(t) = f(t,y)$, $y(t_0) = y_0$ be an IVP.

<u>Thm</u> if $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing (t_0, y_0)

then in some interval $t_0 - h < t < t_0 + h$ inside $\alpha < t < \beta$, there exists a unique solution $y(t)$ to (*).

<u>pf</u> will <u>come</u> later.

Many Comments

(a) $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are functions of 2 variables. Continuity here is bigger than simply continuous in each variable while holding the other variable constant.

(b) $\frac{\partial f}{\partial y}(t,y)$ is the derivative of f with respect to y while pretending t is a const.

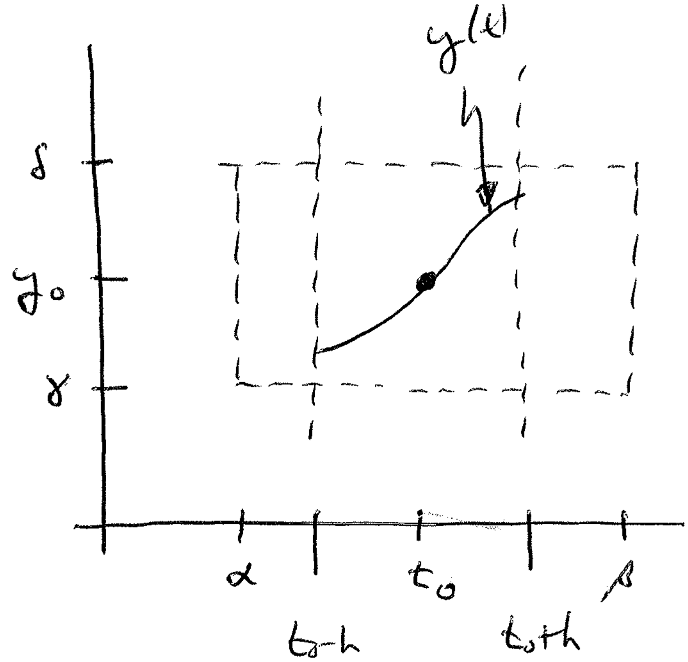
Called a partial derivative

ex: $f(t,y) = t^2 y + \sin(ty)$.

$$\frac{\partial f}{\partial y}(t,y) = t^2 + t\cos(ty).$$

$$\frac{\partial f}{\partial t}(t,y) = 2ty + y\cos(ty).$$

(c) Geometrically, a solution passing through (t_0, y_0) is an integral curve in the ty-plane.



(d) If $f(t,y)$ is continuous near (t_0, y_0), then $y'(t)$ is continuous. But then, by Calc I, $y(t)$ is differentiable. This is enough for solutions to exist through (t_0, y_0).

By the FTC, $y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\, ds$

If $f(t,y)$ is continuous near (t_0, y_0), then this integral will exist. Note:

$$y' = \frac{d}{dt}[y(t)] = \frac{d}{dt}\left[\int_{t_0}^{t} f(s, y(s))\, ds = f(t,y)\right.$$

(e) If $\frac{\partial f}{\partial y}(t,y)$ is continuous near (t_0, y_0), then solutions vary nicely in the y-direction. This is enough to ensure solutions are <u>unique</u>.

$\Rightarrow$ Solution curves never touch or cross when uniquely defined.

(f) <u>Example</u> Given $ty' + 2y = 4t^2$, if we place it in the form $y' = f(t,y)$, we get

$$y' = -\frac{2}{t}y + 4t$$

Here, as long as our initial data does not include $t_0 = 0$, solutions will exist ($-\frac{2}{t}y + 4t$ is cont when $t \neq 0$) and unique ($\frac{\partial f}{\partial y}(t,y) = -\frac{2}{t}$ is cont when $t \neq 0$).

Caution: It may be possible for solutions to exist and/or be unique even at $t=0$. But it is not assured!

Example Given $y' = \frac{3}{2} y^{1/3}$, $f(t,y) = \frac{3}{2} y^{1/3}$.

Here $f(t,y)$ is defined and continuous everywhere (for all $t \in \mathbb{R}$, and $y \in \mathbb{R}$).

Hence by Thm, solutions are guaranteed to exist everywhere.

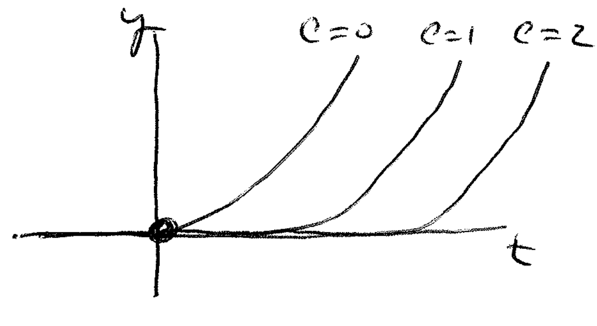
But $\frac{\partial f}{\partial y}(t,y) = \frac{1}{2} y^{-2/3}$ is not continuous along the $y=0$ line (in the ty-plane).

Hence solutions exist for starting values like $y(t_0) = 0$, but may not be unique. (everywhere else they are unique!).

Note: For any value of $c \geq 0$, the curve

$$y(t) = \begin{cases} 0 & t < c \\ (t-c)^{3/2} & t \geq c \end{cases}$$

Solves the IVP $y' = \frac{3}{2} y^{1/3}$, $y(c) = 0$



(9) For linear ODE, existence and uniqueness is easier. In standard form,

$$y' + p(t)\,y = q(t)$$

and in $y' = f(t,y)$ form

$$y' = \underbrace{-p(t)\,y + q(t)}_{f(t,y)}.$$

Thm 2.4.1 says as long as $p(t)$ and $q(t)$ are continuous at t_0, then solutions exist and are unique ~~throug~~ for

$$y' = -p(t)\,y + q(t), \quad y(t_0) = y_0.$$