110.302 Lecture 3: Friday, 9/4/15

Very generally, a first-order ODE of the form

$$\frac{dy}{dt} = f(t,y) \qquad (*)$$

will have f a function of both t and y and will not be solvable.

However, with some additional structure to f, there are methods to solve: In Chapter 2, we explore some of these.

First type of structure (Section 2.1): Linear

(I) Suppose $f(t,y) = -p(t)y + q(t)$ for some ~~[illegible]~~ functions $p(t), q(t)$.

Then $(*)$ can be rewritten

$$y' = -p(t)y + q(t) \quad \text{or}$$

$$(**) \qquad y' + p(t)y = q(t)$$

This new form exposes a structure that facilitates calculation: the LHS is almost the total derivative of a function. To make it so, we multiply the ODE by an expression called an integrating factor

<u>Def</u> An <u>integrating</u> <u>factor</u> is a term that when multiplied to an expression renders the expression integrable.

To understand what we are looking for, look at the patterns here:

Let y be a diff'ble function of t. Then, for any other diff'ble function of t, $f(t)$, we have

$$\frac{d}{dt}\left[f(t)y\right] = f(t)y' + f'(t)y \quad \text{by Prod. Rule}$$

And also $\frac{d}{dt}\left[e^{f(t)}y\right] = e^{f(t)}y' + e^{f(t)}f'(t)y$

$$= e^{f(t)}\left[y' + f'(t)\,y\right].$$

We do this just to look for patterns. In this case, we see an important one: Inside the brackets, $[y' + f'(t)\,y]$ looks very close to the LHS of $(**)$ $y' + p(t)y = q(t)$.

In fact, they are precisely the same when

$$f'(t) = p(t), \quad \text{or} \quad f(t) = \int p(t)\,dt.$$

So we do one more calculation for a pattern:

$$\frac{d}{dt}\left[e^{\int p(t)\,dt}\,y\right] = e^{\int p(t)\,dt}\,y' + \frac{d}{dt}\left[e^{\int p(t)\,dt}\right]y$$

$$= e^{\int p(t)\,dt}\,y' + e^{\int p(t)\,dt}\,p(t)\,y$$

$$= e^{\int p(t)\,dt}\underbrace{\left[y' + p(t)y\right]}_{\substack{\text{precisely the LHS of} \\ (**)\quad y' + p(t)\,y = q(t)}}.$$

This is useful because, if we take $y' + p(t)y = q(t)$ and multiply the entire eqn by $e^{\int p(t)dt}$, then the LHS becomes easily integrable.

Call $e^{\int p(t)dt}$ the integrating factor of $y' + p(t)y = q(t)$.

Challenge Q: It turns out, any antiderivative of $p(t)$ will give the same effect. Why?

Let's play this out and see just how the integrating factor is helpful.

Solve $y' + p(t)y = q(t)$.

Step 1: Multiply entire eqn by $e^{\int p(t)dt}$.

$$e^{\int p(t)dt}\left[y' + p(t)\, y = q(t)\right]$$

$$\underbrace{e^{\int p(t)dt} y' + e^{\int p(t)dt} p(t)\, y}_{\frac{d}{dt}\left[e^{\int p(t)dt} y\right]} = e^{\int p(t)dt} q(t)$$

$$\frac{d}{dt}\left[e^{\int p(t)dt} y\right] = e^{\int p(t)dt} q(t).$$

Step 2: Integrate with respect to (wrt) t.

$$\int \frac{d}{dt}\left[e^{\int p(t)dt} y\right] dt = \int e^{\int p(t)dt} q(t)\, dt$$

$$e^{\int p(t)dt} y = \int e^{\int p(t)dt} q(t)\, dt + C$$

Step 3: Solve for y.

$$y(t) = e^{-\int p(t)dt}\left[\int e^{\int p(t)dt} q(t)\, dt + C\right].$$

Notes ① Theoretically, we can <u>always</u> do this. Practically, the integrating factor $e^{\int p(t)dt}$ is pretty easy to calculate, usually.

② You do not need to memorize anything of the form of step 3. Just remember the steps.

③ Any anti derivative of $p(t)$ will do, since ⓐ they all only differ by a constant ⓑ You are multiplying the entire equation by the factor.

ex. Suppose $p(t) = 2t$. Then $e^{\int p(t)dt} = e^{\int 2t\,dt} = e^{t^2}$.

If instead you chose $e^{\int p(t)dt} = e^{\int 2t\,dt} = e^{t^2+c}$, then

$$e^{t^2+c} = e^{t^2}e^{c} = e^{t^2}K, \text{ for } K \in \mathbb{R} \text{ a constant.}$$

Then $Ke^{t^2}[y' + p(t)y = g(t)]$ is same as $e^{t^2}[y' + p(t)y = g(t)]$ as far as solutions are concerned.

Some examples

(I) Solve $ty' - 2y = t^3 e^{-2t}$

Strategy: This is linear so we use the int. fact. $e^{\int p(t)dt}$ to solve using the 3 steps above.

Solution: Place the ODE in standard form

$$y' - \frac{2}{t}y = t^2 e^{-2t}.$$

This gives us $p(t) = -\frac{2}{t}$, so the int. factor is

$$e^{\int p(t)dt} = e^{-2\int \frac{1}{t}dt} = e^{-2\ln|t|} = e^{\ln t^{-2}} = t^{-2}.$$

Step 1: Multiply ODE by int. factor.

$$t^{-2}\left[y' - \frac{2}{t}y = t^2 e^{-2t}\right]$$

$$\underbrace{t^{-2}y' - \frac{2}{t^3}y}_{} = e^{-2t}$$

$$\frac{d}{dt}\left[t^{-2}y\right] = e^{-2t}$$

Step 2: Integrate wrt t.

$$\int \frac{d}{dt}\left[t^{-2}y\right]dt = t^{-2}y + C_1 = \int e^{-2t}dt = -\frac{1}{2}e^{-2t} + C_2$$

$$t^{-2}y = -\frac{1}{2}e^{-2t} + K$$

Step 3: Solve for $y(t)$:

$$\boxed{y(t) = -\frac{1}{2}t^2 e^{-2t} + Kt^2}$$

This func solves the ODE.

Check to see if this is correct:

$$\underbrace{\left(-te^{-2t}+t^2e^{-2t}+2Kt\right)}_{y'}-\frac{2}{t}\underbrace{\left(-\frac{1}{2}t^2e^{-2t}+Kt^2\right)}_{y}=t^2e^{-2t}$$

$$\underbrace{-te^{-2t}+t^2e^{-2t}+\underbrace{2Kt+te^{-2t}}_{0}-2Kt}_{0}=t^2e^{-2t}$$

$$t^2e^{-2t}=t^2e^{-2t} \quad \checkmark$$

(II) $\dot{x}+2tx=t^3$. Solve this.

<u>Strategy</u>: Use the integrating factor on this linear ODE to integrate through to an expression for $x(t)$.

<u>Solution</u>: This ODE is linear, with $p(t)=2t$.

Thus the int. factor is

$$e^{\int p(t)dt}=e^{\int 2t\,dt}=e^{t^2}.$$

<u>Step 1</u>: Mult. ODE by int. factor:

$$e^{t^2}\left[\dot{x}+2tx=t^3\right]$$

$$\underbrace{e^{t^2}\dot{x}+2te^{t^2}x}=t^3e^{t^2}$$

$$\frac{d}{dt}\left[e^{t^2}x\right]=t^3e^{t^2}.$$

Step 2: Integrate wrt t.

$$\int \frac{d}{dt}\left[e^{t^2}x\right]dt = e^{t^2}x + C_1 = \underbrace{\int t^3 e^{t^2}\,dt}$$

$$\int t^3 e^{t^2}\,dt \;\xrightarrow[ds=2t\,dt]{\text{subst. } s=t^2}\; \underbrace{\frac{1}{2}\int s e^s\,ds}$$

$$\frac{1}{2}\int s e^s\,ds \;\xrightarrow[\substack{du=ds\\ dv=e^s ds}]{\substack{\text{Int By Pts.}\\ u=s\\ v=e^s}}\; \frac{1}{2}\left(se^s - \int e^s\,ds\right) = \frac{1}{2}\left(se^s - e^s\right) + C_2$$

$$= \frac{1}{2}e^{t^2}(t^2-1) + C_2$$

Combine constants to get:

$$e^{t^2}x = \frac{1}{2}e^{t^2}(t^2-1) + K$$

Step 3: Solve for $x(t)$.

$$\boxed{x(t) = \frac{1}{2}t^2 - \frac{1}{2} + Ke^{-t^2}}$$

This is the general solution to $\dot{x} + 2tx = t^3$

Check this:

$$\underbrace{\left(t - 2Kte^{-t^2}\right)}_{\dot{x}} + 2t\underbrace{\left(\frac{1}{2}t^2 - \frac{1}{2} + Ke^{-t^2}\right)}_{x} = t^3$$

$$\cancel{t} - 2K\cancel{t}e^{-t^2} + t^3 - \cancel{t} + 2K\cancel{t}e^{-t^2} = t^3$$

$0 \qquad 0$

$$t^3 = t^3$$

it works

III Solve $\frac{dx}{ds} = \frac{x}{s} - s^2$, for $s > 0$

Here the ODE is again linear (note s is the independent variable), and $p(s) = -\frac{1}{s}$.

The int. factor is then

$$e^{\int p(s)ds} = e^{\int(-\frac{1}{s})ds} = e^{-\int \frac{1}{s} ds} = e^{-\ln s} = e^{\ln s^{-1}} = s^{-1}$$

Multiply through standard form of ODE to get

$$\frac{1}{s}\left[\frac{dx}{ds} - \frac{x}{s} = -s^2\right] \Rightarrow \underbrace{\frac{1}{s}\frac{dx}{ds} - \frac{x}{s^2}}_{\frac{d}{ds}\left[\frac{1}{s}\cdot x\right]} = -s$$

$$\frac{d}{ds}\left[\frac{1}{s}\cdot x\right] = -s$$

Integrate wrt s to get

$$\frac{1}{s}\cdot x = \int(-s)ds + C = -\frac{s^2}{2} + C$$

Solve for $x(s)$:

$$\boxed{x(s) = -\frac{s^3}{2} + Cs}$$

This is the general solution to ODE

Check it:

$$\underbrace{\left(-\frac{3}{2}s^2 + C\right)}_{\frac{dx}{ds}} = \frac{1}{s}\underbrace{\left(-\frac{s^3}{2} + Cs\right)}_{x} - s^2$$

$$-\frac{3}{2}s^2 + C = -\frac{s^2}{2} + C - s^2$$

$$-\frac{3}{2}s^2 = -\frac{3}{2}s^2 \checkmark$$ It is correct.

IV Find the general solution to

$$t(y'-y) = (1+t^2)e^t \quad \text{on } t>0.$$

Here, try to see why this is linear, with $p(t) = -1$. The solution is

$$y(t) = e^t\left(\ln t + \frac{t^2}{2} + c\right)$$

This solution is drawn up in a separate document under example problems. on the web site.

V Solve $\frac{dp}{dt} = \frac{p}{2} - 450$ using an integrating factor.

Solution: This is an exercise. You already know the answer.